Lektion 16

$\boxed{P.5}$

15 a) Partialbråksuppdelar!

$$\frac{x}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\frac{x}{(x+1)^3} = \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

$$x = Ax^2 + 2Ax + A + Bx + B + C$$

$$x = Ax^2 + (2A+B)x + (A+B+C)$$

$$\Rightarrow \quad A = 0 \qquad B = 1$$
$$\cancel{2A} + B = 1 \qquad C = -1$$
$$\cancel{A} + B + C = 0$$

$$\Rightarrow \int \frac{x}{(x+1)^3}\,dx = \int \frac{1}{(x+1)^2}\,dx + \int \frac{-1}{(x+1)^3}\,dx$$

$$= \frac{(x+1)^{-1}}{-1} - \frac{(x+1)^{-2}}{-2} + C =$$

$$= -\frac{1}{x+1} + \frac{1}{2(x+1)^2} + C.$$

b) $$\frac{x^2}{x^4 - 8x^2 + 16} = \frac{x^2}{(x^2-4)^2} = \frac{x^2}{(x-2)^2(x+2)^2}$$

$$\frac{x^2}{(x-2)^2(x+2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

$$x^2 = A(x-2)(x+2)^2 + B(x+2)^2 + C(x+2)(x-2)^2$$
$$+ D(x-2)^2$$

$$x^2 = A(x+2)(x^2-4) + B(x+2)^2 + C(x-2)(x^2-4)$$
$$+ D(x-2)^2$$

$$x^2 = Ax^3 + 2Ax^2 - 4Ax - 8A + Bx^2 + 4Bx + 4B$$
$$+ Cx^3 - 2Cx^2 - 4Cx + 8C + Dx^2 - 4Dx + 4D$$

$$x^2 = \overbrace{(A+C)}^{=0}x^3 + \overbrace{(2A - 2C + B + D)}^{=1}x^2$$
$$+ (\underbrace{-4A + 4B - 4C - 4D}_{=0})x + (\underbrace{-8A + 4B + 8C + 4D}_{=0})$$

$C = -A$

$4A + B + D = 1$

$4B - 4D = 0 \Rightarrow B = D$

$-16A + 8B = 0 \Rightarrow B = 2A$

$B = 2A$

$C = -A$

$D = 2A$

$4A + 2A + 2A = 1$

$$\Rightarrow A = \frac{1}{8},\quad B = \frac{1}{4},\quad C = -\frac{1}{8},\quad D = \frac{1}{4}$$

$$\Rightarrow \int \frac{x^2}{x^4 - 8x^2 + 16}\,dx = \frac{1}{8}\int \frac{dx}{x-2} + \frac{1}{4}\int \frac{dx}{(x-2)^2} - \frac{1}{8}\int \frac{dx}{x+2}$$
$$+ \frac{1}{4}\int \frac{dx}{(x+2)^2} =$$
$$= \frac{1}{8}\ln|x-2| - \frac{1}{4(x-2)} - \frac{1}{8}\ln|x+2|$$
$$- \frac{1}{4(x+2)} + C.$$

c) Partialbråkuppdelar: $(x-1)^2 = (x-1)^2(x+1)^2 \Rightarrow$

$$\frac{1}{(x^2-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$1 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1)$$
$$+ D(x-1)^2$$

$$1 = A(x^2-1)(x+1) + B(x+1)^2 + C(x^2-1)(x-1)$$
$$+ D(x-1)^2$$

$$1 = Ax^3 - Ax + Ax^2 - A + Bx^2 + 2Bx + B + Cx^3 - Cx - Cx^2 + C + Dx^2 - 2Dx + D$$

$$1 = \underbrace{(A+C)}_{=0}x^3 + \underbrace{(A+B-C+D)}_{=0}x^2 + \underbrace{(-A+2B-C-2D)}_{=0}x + \underbrace{(-A+B+C+D)}_{=1}$$

$C = -A$

$2A + B + D = 0$

$\cancel{-A} + 2B + A - 2D = 0$

$-2A + B + D = 1$

$C = -A$

$B = D$

$2A + 2B = 0 \Rightarrow B = -A$

$-2A - 2A = 1 \Rightarrow A = -\frac{1}{4}$

$$A = -\frac{1}{4},\quad B = \frac{1}{4},\quad C = \frac{1}{4},\quad D = \frac{1}{4}$$

$$\Rightarrow \int \frac{1}{(x^2-1)^2}\,dx = -\frac{1}{4}\int\frac{dx}{x-1} + \frac{1}{4}\int\frac{dx}{(x-1)^2} + \frac{1}{4}\int\frac{dx}{x+1} + \frac{1}{4}\int\frac{dx}{(x+1)^2} =$$

$$= -\frac{1}{4}\ln|x-1| - \frac{1}{4(x-1)} + \frac{1}{4}\ln|x+1| - \frac{1}{4(x+1)} + C$$

e) Partielbråkuppdelar:

$$\frac{1}{x^3+2x^2+5x} = \frac{1}{x\underbrace{(x^2+2x+5)}_{\text{saknar rötter}}} = \frac{A}{x} + \frac{Bx+D}{x^2+2x+5}$$

$$= \frac{Ax^2 + 2Ax + 5A + Bx^2 + Dx}{x(x^2+2x+5)}$$

$$1 = (A+B)x^2 + (2A+D)x + 5A \Rightarrow \begin{array}{l} A = 1/5 \\ B = -1/5 \\ D = -2/5 \end{array}$$

$$\int \frac{dx}{x^3+2x^2+5x} = \frac{1}{5}\int\frac{dx}{x} - \frac{1}{5}\underbrace{\int\frac{x+2}{x^2+2x+5}\,dx}_{=I} = \otimes$$

$$I = \int \frac{x+2}{(x+1)^2+4}\,dx = \begin{bmatrix} x+1=t \\ dx=dt \end{bmatrix} =$$

$$= \frac{1}{4}\cdot\frac{1}{(\frac{t}{2})^2+1}$$

$$= \int \frac{t+1}{t^2+4}\,dt = \int \frac{t}{t^2+4}\,dt + \int\frac{dt}{t^2+4} =$$

Var.byte $t^2+4=u$

$$= \frac{1}{2}\ln(t^2+4) + \frac{1}{2}\arctan\frac{t}{2} + C =$$

$$= \frac{1}{2}\ln((x+1)^2+4) + \frac{1}{2}\arctan\frac{x+1}{2} + C$$

$$\otimes = \frac{1}{5}\ln|x| - \frac{1}{10}\ln(x^2+2x+5) - \frac{1}{10}\arctan\frac{x+1}{2} + C$$

f) $$\frac{2x}{(x^2+4)(x^2+2x+4)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+2x+4}$$

Inga rötter

$$2x = (Ax+B)(x^2+2x+4) + (Cx+D)(x^2+4)$$

$$2x = Ax^3 + Bx^2 + 2Ax^2 + 2Bx + 4Ax + 4B + Cx^3 + Dx^2 + 4Cx + 4D$$

$$2x = \underbrace{(A+C)}_{=0}x^3 + \underbrace{(B+2A+D)}_{=0}x^2 + \underbrace{(2B+4A+4C)}_{=2}x + \underbrace{(4B+4D)}_{=0}$$

$C = -A$

$B = -D$

$\cancel{B} + 2A + \cancel{D} = 0 \Rightarrow A = 0 = C$

$2B + \cancel{4A} + \cancel{4A} = 2 \Rightarrow B = 1$

$D = -1$

$$\int \frac{2x}{(x^2+4)(x^2+2x+4)}dx = \int \frac{dx}{x^2+4} - \int \frac{dx}{x^2+2x+4}$$

$$= \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} - \int \frac{dx}{(x+1)^2+(\sqrt{3})^2}$$

$$= \frac{1}{\cancel{4}_{2}} \cdot \cancel{2} \arctan\frac{x}{2} - \frac{1}{\sqrt{3}} \arctan\frac{x+1}{\sqrt{3}} + C.$$

g) Polynomdivision!

$$\begin{array}{l|l} \multicolumn{2}{l}{x^2} \\ x^5+x^3-5x^2-3x+12 & x^3+x-10 \\ -\;x^5+x^3-10x^2 & \\ \hline \quad 5x^2-3x+12 & \end{array}$$

$$\int \ldots dx = \int x^2 dx + \underbrace{\int \frac{5x^2-3x+12}{x^3+x-10}dx}_{=I}$$

Observera att $x^3+x-10=0$ har en rot $x=2$:

$$\begin{array}{l|l} \multicolumn{2}{l}{x^2+2x+5} \\ x^3+x-10 & x-2 \\ -\;x^3-2x^2 & \\ \hline \quad -\;2x^2+x-10 & \\ \quad 2x^2-4x & \\ \hline \qquad -\;5x-10 & \\ \qquad 5x-10 & \\ \hline \qquad\quad 0 & \end{array}$$

$$\Rightarrow x^3+x-10 = (x-2)\underbrace{(x^2+2x+5)}_{\text{inga rötter}}$$

Partialbråkuppdelning!

$$\frac{5x^2-3x+12}{(x-2)(x^2+2x+5)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+5}$$

$$5x^2-3x+12 = Ax^2+2Ax+5A+Bx^2+Cx-2Bx-2C$$

$$5x^2-3x+12 = (A+B)x^2+(2A+C-2B)x+5A-2C$$

$A + B = 5$ $\qquad$ $B = 5 - A$

$2A + C - 2B = 3$ $\qquad$ $C = \dfrac{-12 + 5A}{2}$

$5A - 2C = 12$ $\qquad$ $2A - 6 + \frac{5}{2}A - 10 + 2A = 3$

$$\frac{13A}{2} = +13 \Rightarrow A = +2$$
$$B = 3$$
$$C = -1$$

$$I = +2\int \frac{dx}{x-2} + \int \frac{3x-1}{x^2+2x+5}\,dx =$$

$$= +2\ln|x-2| + \underbrace{\int \frac{3x-1}{(x+1)^2+2^2}\,dx}_{I_1} = \otimes$$

$$I_1 = \begin{bmatrix} x+1 = t \\ dx = dt \end{bmatrix} = \int \frac{3t-4}{t^2+2^2}\,dt =$$

$$= 3\int \frac{t}{t^2+2^2}\,dt - 4\int \frac{dt}{t^2+2^2} =$$

$$= \frac{3}{2}\ln(t^2+4) - 2\arctan\frac{t}{2} + C =$$

$$= \frac{3}{2}\ln(x^2+2x+5) - 2\arctan\frac{x+1}{2} + C$$

Svar:

$$\frac{x^3}{3} + 2\ln|x-2| + \frac{3}{2}\ln(x^2+2x+5) - 2\arctan\frac{x+1}{2} + C$$

a) $\displaystyle\int \frac{\ln(1+x^2)}{x^3}\,dx = \left[\begin{array}{ll} f(x) = x^{-3} & F(x) = -\frac{1}{2x^2} \\ g(x) = \ln(1+x^2) & g'(x) = \frac{2x}{1+x^2} \end{array}\right]$

$$= -\frac{\ln(1+x^2)}{2x^2} + \underbrace{\int \frac{dx}{x(1+x^2)}}_{=I} = \otimes$$

I: $\displaystyle\frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{1}{x(1+x^2)}$

$A + Ax^2 + Bx^2 + Cx = 1 \Rightarrow$ $A = 1$

$B = -A = -1$

$C = 0$

$$I = \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx = \ln|x| - \frac{1}{2}\ln|1+x^2| + C$$

$$\otimes = -\frac{\ln(1+x^2)}{2x^2} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$$

b) $\displaystyle\int \frac{x \arctan x}{(x^2+2)^2}\,dx = \left[\begin{array}{l} f'(x) = \frac{x}{(x^2+2)^2},\ F(x) = \frac{-1}{2(x^2+2)} \\ g(x) = \arctan x,\ g'(x) = \frac{1}{1+x^2} \end{array}\right]$

$$= -\frac{\arctan x}{2(x^2+2)} + \underbrace{\int \frac{1}{2(x^2+1)(x^2+2)}\,dx}_{I} = \otimes$$

$$I = \frac{1}{2}\int\left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right)dx =$$

$$= \frac{1}{2}\arctan x - \frac{1}{2\sqrt{2}}\arctan\frac{x}{\sqrt{2}}$$

$$\otimes = -\frac{\arctan x}{2(x^2+2)} + \frac{1}{2}\arctan x - \frac{1}{2\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C$$

19

a) $\displaystyle\int \frac{e^{x}dx}{e^{x}+e^{-x}} = \int \frac{e^{x}\,dx}{e^{2x}+1} = \begin{bmatrix} e^{x}=t \\ dt=e^{x}dx \end{bmatrix} =$

$\displaystyle = \int \frac{dt}{t^2+1} = \arctan t + C = \arctan e^{x} + C$

b) $\displaystyle\int \frac{dx}{x(1+x^n)} = \begin{bmatrix} 1+x^n = t \\ dt = nx^{n-1}dx \end{bmatrix} \overset{\text{antar } n\neq 0}{=}$

$\displaystyle = \int \frac{nx^{n-1}\,dx}{n\underbrace{x^n}_{t-1}(1+x^n)} = \int \frac{dt}{n(t-1)\,t}$

$\displaystyle = \frac{1}{n}\int \left(\frac{1}{t-1} - \frac{1}{t}\right)dt =$

$\displaystyle = \frac{1}{n}\left(\ln|t-1| - \ln|t|\right) + C =$

$\displaystyle = \frac{1}{n}\left(\ln x^n - \ln|x^n+1|\right) + C \overset{x>0}{=}$

$\displaystyle = \frac{1}{\cancel{n}}\cdot\cancel{n}\ln x - \frac{1}{n}\ln(x^n+1) + C$

om $n \neq 0$

Om $n=0 \Rightarrow \displaystyle\int \frac{dx}{2x} = 2\ln x + C$

c) $\displaystyle\int x\left((\ln x)^2 + e^{-2x}\right)dx = \underbrace{\int x(\ln x)^2dx}_{=I_1} + \underbrace{\int x e^{-2x}dx}_{=I_2}$

$I_1 = \begin{bmatrix} \ln x = t \Rightarrow x = e^{t} \\ dx = e^{t}dt \end{bmatrix} =$

$\displaystyle = \int e^{t}\cdot t^2 e^{t}dt = \int t^2 e^{2t}dt =$

$\displaystyle = \frac{1}{2}e^{2t}\cdot t^2 - \int \cancel{2}t\frac{1}{\cancel{2}}e^{2t}dt =$

$\displaystyle = \frac{1}{2}t^2e^{2t} - \cancel{2}\left(\frac{1}{2}e^{2t}\cdot t - \int\frac{1}{2}e^{2t}dt\right) =$

$\displaystyle = \frac{1}{2}t^2e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} =$

$$= \frac{1}{2}(\ln x)^2 \cdot x^2 - \frac{1}{2}\ln x \cdot x^2 + \frac{1}{4}x^2$$

$$I_2 = \int x \cdot e^{-2x}\,dx = -\frac{1}{2}e^{-2x} \cdot x + \int \frac{1}{2}e^{-2x}\,dx$$

$$= -\frac{1}{2}e^{-2x} \cdot x - \frac{1}{4}e^{-2x}$$

$$I_1 + I_2 = \frac{x^2}{4}\left(1 + 2(\ln x)^2 - 2\ln x\right) - \frac{1}{4}e^{-2x}(1+2x) + C$$

d) $\int x\left(\ln(x^2+1) - x^2\right)dx = \left[\begin{array}{ll} x^2+1 = t & x^2 = t-1 \\ dt = 2x\,dx & \end{array}\right]$

$$= \frac{1}{2}\int \left(\ln(x^2+1) - x^2\right)\underbrace{2x\,dx}_{dt} =$$

$$= \frac{1}{2}\int (\ln t - t + 1)\,dt = \otimes$$

$$\int \ln t \cdot 1\,dt = t \cdot \ln t - \int t \cdot \frac{1}{t}\,dt =$$

$$= t\ln t - t$$

$$\otimes = \frac{1}{2}\left(t\ln t - \cancel{t} - \frac{t^2}{2} + \cancel{t}\right) + C =$$

$$= \frac{1}{2} \cdot (x^2+1)\ln(x^2+1) - \frac{(x^2+1)^2}{4} + C$$

Extra
[P5]

33

$$\frac{ax^2+bx+c}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

OBS! Rationell funktion kan skrivas på formen $\frac{\text{polynom}}{\text{polynom}}$.

Om $A \neq 0$ och $D \neq 0$ $\Rightarrow$ då får vi efter integrering logaritmiska termer då $\int \frac{A}{x}\,dx = A \ln|x|$ osv.

Så $A = D = 0$ $\Rightarrow$

$$\frac{ax^2+bx+c}{x^3(x-1)^2} = \frac{B}{x^2} + \frac{C}{x^3} + \frac{E}{(x-1)^2}$$

$$ax^2+bx+c = Bx(x-1)^2 + C(x-1)^2 + Ex^3$$

$$ax^2+bx+c = Bx^3 - 2Bx^2 + Bx + Cx^2 - 2Cx + C + Ex^3$$

$$= (B+E)x^3 + (-2B+C)x^2 + (B-2C)x + C$$

$$\left.\begin{aligned} c &= C \\ b &= B-2C \\ a &= C-2B \\ E &= -B \end{aligned}\right) \Rightarrow \begin{aligned} a &= c-2B \\ b &= B-2c \Rightarrow B = b+2c \end{aligned}$$

$$\Rightarrow a = c - 2b - 4c$$

eller $a + 2b + 3c = 0$

15 h) se boken s. 261;

gör part. integration på

$$\int \frac{1}{x^2+2} \cdot 1 \, dx = \frac{x}{x^2+2} + \int \frac{2x^2}{(x^2+2)^2} dx =$$

$$= \frac{x}{x^2+2} + \int \frac{2x^2+4-4}{(x^2+2)^2} dx =$$

$$= \frac{x}{x^2+2} + 2\int \frac{dx}{x^2+2} - 4\int \frac{dx}{(x^2+2)^2}$$

$$\Rightarrow 4\int \frac{dx}{(x^2+2)^2} = \frac{x}{x^2+2} + \int \frac{dx}{x^2+2} \Rightarrow$$

$$\int \frac{dx}{(x^2+2)^2} = \frac{x}{4(x^2+2)} + \frac{1}{4} \cdot \int \frac{dx}{x^2+\underbrace{2}_{(\sqrt{2})^2}} =$$

$$= \frac{x}{4(x^2+2)} + \frac{1}{4\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + C$$