Lektion 20

$\boxed{B6}$ **1** Φ är en trappfunktion $\Rightarrow$ man kan använda definition 6.1. Alternativt kan man beräkna

$$\int_{-2}^{7} \Phi(x)\,dx = \int_{-2}^{2} 3\,dx + \int_{2}^{2} 5\,dx + \int_{2}^{5} (-5)\,dx + \int_{5}^{7} (-1)\,dx =$$

$$= \left[3x\right]_{-2}^{2} + \left[5x\right]_{+2}^{2} + \left[-5x\right]_{2}^{5} + \left[-x\right]_{5}^{7} =$$

$$= 3(2+2) + \cancel{5(2-2)} - 5(5-2) - (7-5) =$$

$$= 12 - 15 - 2 = -5$$

11 e) se exempel 6.12. Funktionen $f(x) = \frac{x}{\cos^2 x}$ är udd eftersom

$$f(-x) = \frac{-x}{\cos^2(-x)} = -\frac{x}{\cos^2 x} = -f(x)$$ och

intervallet $\left(-\frac{\pi}{5}; \frac{\pi}{5}\right)$ är symmetrisk i origo $\Rightarrow$

$$\int_{-\frac{\pi}{5}}^{\frac{\pi}{5}} \frac{x}{\cos^2 x}\,dx = 0.$$

Kolla hur beviset går till i Exempel 6:12!

f) $$\int_0^1 \arcsin^2 x\,dx = \left[\begin{array}{l} \text{var. byte:} \\ \arcsin x = t : [0; 1] \to [0; \tfrac{\pi}{2}] \\ x = \sin t \Rightarrow dx = \cos t\,dt \end{array}\right]$$

$$= \int_0^{\frac{\pi}{2}} t^2 \cos t\,dt = \left[t^2 \sin t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2t)\sin t\,dt =$$

$$= \frac{\pi^2}{4} - 2\left(\left[-t\cos t\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t\,dt\right) =$$

$$= \frac{\pi^2}{4} - 2\left((0-0) + \left[\sin t\right]_0^{\frac{\pi}{2}}\right) = \frac{\pi^2}{4} - 2.$$

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$\boxed{P6}$ **4** Använder analysens huvudsats:

Om $S(x) = \int_a^x f(t)\,dt \Rightarrow S'(x) = f(x)$

a) $f(x) = \int_0^x t^2 \ln(t+1)\,dt \Rightarrow \underline{f'(x) = x^2 \ln(x+1)}$

b) $f(x) = \int_x^1 \frac{t^4}{t^2+1}\,dt = -\int_1^x \frac{t^4}{t^2+1}\,dt \Rightarrow$

$f'(x) = -\left(\int_1^x \frac{t^4}{t^2+1}\,dt\right)' = -\underline{\frac{x^4}{x^2+1}}$

c) Det är klart att $f(x) = \int_0^1 e^{t^2}\,dt = \text{konst} \Rightarrow$

$\Rightarrow \underline{f'(x) = 0}$

d) $f(x) = \int_x^{x^2} \frac{e^t}{t}\,dt = \int_x^1 \frac{e^t}{t}\,dt + \int_1^{x^2} \frac{e^t}{t}\,dt =$

$$= \underbrace{-\int_1^x \frac{e^t}{t}\,dt}_{g(x)} + \underbrace{\int_1^{x^2} \frac{e^t}{t}\,dt}_{h(x)} = g(x) + h(x)$$

$g'(x) = -\frac{e^x}{x}$, $h(x)$ är mer komplicerad!

$h(x) = F(x^2)$ där $F(x) = \int_1^x \frac{e^t}{t}\,dt \Rightarrow$

$F'(x) = \frac{e^x}{x}$. Det betyder att $h(x)$ ska deriveras enligt kedjeregeln:

$h'(x) = F'(x^2)\cdot(x^2)' = 2x\,F'(x^2) =$

$= 2x \cdot \frac{e^{x^2}}{x^2} = \underline{\frac{2e^{x^2}}{x}}$.

Slutligen,

$f'(x) = g'(x) + h'(x) = -\frac{e^x}{x} + \frac{2e^{x^2}}{x} = \underline{\frac{2e^{x^2} - e^x}{x}}$

$\underline{\underline{10}}$ a) $\int_0^{1/2} \frac{\arctan 2x}{(1-x)^2}dx$ – beräknar först en primitiv funktion:

$$\int \overset{\downarrow}{\arctan 2x} \cdot \overset{\uparrow}{\frac{1}{(1-x)^2}}\, dx = \left[\int \frac{dx}{(1-x)^2} = +(1-x)^{-1} \right]$$

$$= +(1-x)^{-1}\arctan 2x - \int \frac{2}{(1-x)(1+4x^2)}\, dx =$$

$$= \frac{\arctan 2x}{1-x} + 2\underbrace{\int \frac{dx}{(x-1)(1+4x^2)} dx}_{=I} = \otimes$$

Partialbråkuppdelar:

$$\frac{1}{(x-1)(1+4x^2)} = \frac{A}{x-1} + \frac{Bx+C}{4x^2+1} \Rightarrow$$

$$1 = 4Ax^2 + A + Bx^2 + Cx - Bx - C$$

$$1 = (4A+B)x^2 + (C-B)x + A - C$$

$$\Rightarrow \begin{array}{lll} 4A+B=0 & B=C & B=-\frac{4}{5} \\ C-B=0 & A=C+1 & A=\frac{1}{5} \\ A-C=1 & 4C+4+C=0 \Rightarrow & C=-\frac{4}{5} \end{array}$$

$$\Rightarrow I = \int \left(\frac{1}{5(x-1)} - \frac{4}{5}\, \frac{x+1}{1+4x^2} \right) dx =$$

$$= \frac{1}{5}\ln|x-1| - \frac{4}{5}\int \frac{x\,dx}{1+4x^2} - \frac{4}{5}\int \frac{1}{1+4x^2}\,dx =$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln(1+4x^2) - \frac{2}{5}\arctan 2x$$

$$\otimes = -\frac{\arctan 2x}{x-1} + \frac{2}{5}\ln|x-1| - \frac{1}{5}\ln(1+4x^2) - \frac{4}{5}\arctan 2x$$

$$\Rightarrow \int_0^{1/2} \frac{\arctan 2x}{(1-x)^2}dx = \frac{\arctan 1}{\frac{1}{2}} + \frac{2}{5}\ln\frac{1}{2} - \frac{1}{5}\ln 2 - \frac{4}{5}\cdot\frac{\pi}{4} =$$

$$= \overbrace{2\cdot\frac{\pi}{4}}^{=+\frac{\pi}{2}} - \frac{2}{5}\ln 2 - \frac{1}{5}\ln 2 - \frac{\pi}{5} =$$

$$= \frac{3\pi}{10} - \frac{3}{5}\ln 2$$

d) $\int \tan^2 x \, dx = \left[\begin{array}{ll} \tan x = t & \\ x = \arctan t & dx = \frac{dt}{1+t^2} \end{array} \right] =$

$= \int t^2 \frac{dt}{1+t^2} = \int \frac{t^2+1-1}{t^2+1} dt = \int 1 - \frac{1}{t^2+1} dt =$

$= t - \arctan t = \tan x - x \Rightarrow$

$\int_0^{\pi/4} \tan^2 x \, dx = \left[\tan x - x \right]_0^{\pi/4} = 1 - \frac{\pi}{4}.$

f) $\int \frac{\sin 2x}{(1+\cos x)^3} dx = \int \frac{2\cos x \cdot \sin x}{(1+\cos x)^3} dx = \left[\begin{array}{l} \cos x = t \\ dt = -\sin x \, dx \end{array} \right]$

$= \int \frac{2t}{(1+t)^3} (-dt) = -2 \int \frac{t}{(1+t)^3} dt = \otimes$

Partialbråkuppdelning:

$\frac{t}{(1+t)^3} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{C}{(1+t)^3}$

Handpåläggning $\Rightarrow$ $C = -1$, vidare

$(*)$ $\underbrace{\frac{t}{(1+t)^2}}_{\to 0} = A + \underbrace{\frac{B}{(1+t)}}_{\to 0} + \underbrace{\frac{C}{(1+t)^2}}_{\to 0} \Rightarrow A = 0$ då $t \to \infty$

Låt nu $t = 0$ i $(*)$: $0 = 0 + B + C \Rightarrow B = -C = 1$

$\Rightarrow \otimes = -2 \int \left(\frac{1}{(1+t)^2} - \frac{1}{(1+t)^3} \right) dt = -2 \left(-(1+t)^{-1} + \frac{1}{2}(1+t)^{-2} \right)$

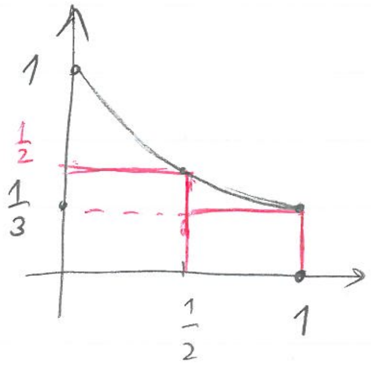
$= \frac{2}{1+\cos x} - \frac{1}{(1+\cos x)^2}$

$\Rightarrow \int_0^{\frac{\pi}{3}} \frac{\sin 2x}{(1+\cos x)^3} dx = \frac{2}{1+\frac{1}{2}} - \frac{1}{(1+\frac{1}{2})^2} - \frac{2}{2} + \frac{1}{4} =$

$= \frac{4}{3} - \frac{4}{9} - 1 + \frac{1}{4} = \frac{48-16-36+9}{36} = \frac{5}{36}$

2 Funktionen $f(x)=\sqrt{\dfrac{3}{3+16x+8x^3}}$ är monoton avtagande i $[0;1]$,

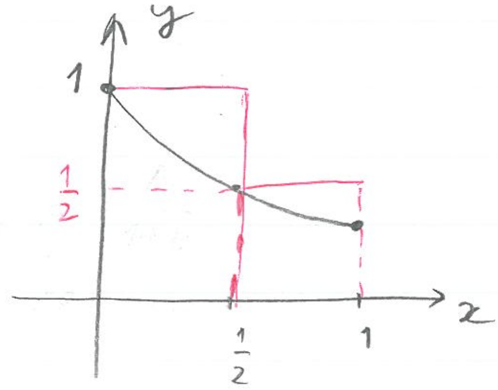
$f(0)=1,\quad f(1)=\sqrt{\dfrac{3}{27}}=\dfrac{1}{3},\quad f\left(\dfrac{1}{2}\right)=\sqrt{\dfrac{3}{12}}=\dfrac{1}{2}$



Betrakta undertappan på bilden. Detta ger att

$$\int_0^1 f(x)\,dx \ge \frac{1}{2}\cdot\left(\frac{1}{2}-0\right)+\frac{1}{3}\left(1-\frac{1}{2}\right)=$$

$$=\frac{1}{4}+\frac{1}{6}=\frac{5}{12}>\frac{1}{3}$$



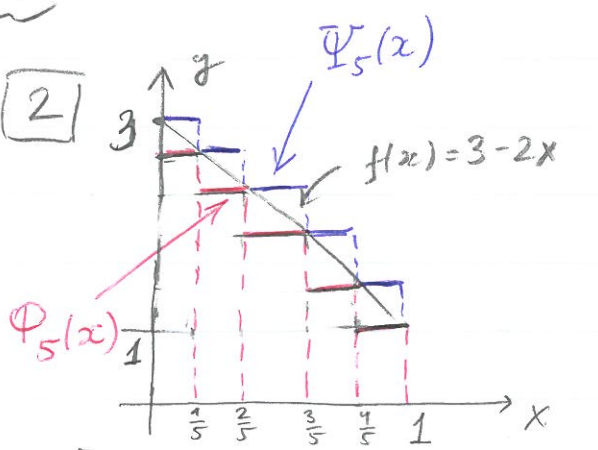
Betrakta nu övertrappan på bilden. Detta ger att

$$\int_0^1 f(x)\,dx \le 1\cdot\left(\frac{1}{2}-0\right)+\frac{1}{2}\left(1-\frac{1}{2}\right)=$$

$$=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}.$$

Extra

B6 $\boxed{2}$



a) $\displaystyle\int_0^1 \Phi_5(x)\,dx=$ $\boxed{5}$

$$=\frac{1}{5}\cdot\left(3-\frac{2}{5}\right)+\frac{1}{5}\left(3-\frac{4}{5}\right)+\frac{1}{5}\left(3-\frac{6}{5}\right)+\frac{1}{5}\left(3-\frac{8}{5}\right)+\frac{1}{5}\left(3-\frac{10}{5}\right)$$

$$=\frac{1}{5}\left(15-\frac{2+4+6+8+10}{5}\right)$$

$$=\frac{1}{5}\left(15-\frac{30}{5}\right)=\frac{1}{5}\cdot 9=\frac{9}{5}$$

$$\int_0^1 \Psi_5(x)\,dx=\frac{1}{5}\left[\left(3-0\right)+\left(3-\frac{2}{5}\right)+\left(3-\frac{4}{5}\right)+\left(3-\frac{6}{5}\right)+\left(3-\frac{8}{5}\right)\right]=$$

$$=\frac{1}{5}\left(15-\frac{2+4+6+8}{5}\right)=\frac{1}{5}\cdot\left(15-\frac{20}{5}\right)=\frac{11}{5}$$

b) Likadant finner vi att

$$\int_0^1 \Phi_{10}(x)\,dx = \frac{1}{10}\left[f\left(\frac{1}{10}\right) + f\left(\frac{2}{10}\right) + \dots + f\left(\frac{10}{10}\right)\right] =$$

$$= \frac{1}{10}\left[\left(3-\frac{2}{10}\right) + \left(3-\frac{4}{10}\right) + \dots + \left(3-\frac{20}{10}\right)\right] =$$

$$= \frac{1}{10}\left[30 - \frac{2+4+6+\dots+20}{10}\right] =$$

$$= 3 - \frac{1}{10}\cdot\frac{1+2+3+\dots+10}{5} = 3 - \frac{\frac{1+10}{2}\cdot\cancel{10}}{\cancel{50}\ 5} =$$

$$= 3 - \frac{11}{10} = \frac{30-11}{10} = \frac{19}{10}$$, och

$$\int_0^1 \Psi_{10}(x)\,dx = \frac{1}{10}\left[f(0) + f\left(\frac{1}{10}\right) + f\left(\frac{2}{10}\right) + \dots + f\left(\frac{9}{10}\right)\right] =$$

$$= \frac{1}{10}\left[(3-0) + \left(3-\frac{2}{10}\right) + \left(3-\frac{4}{10}\right) + \dots + \left(3-\frac{18}{10}\right)\right] =$$

$$= \frac{1}{10}\left[30 - \frac{2+4+\dots+18}{10}\right] = 3 - \frac{1}{10}\cdot\frac{1+2+\dots+9}{5} =$$

$$= 3 - \frac{\frac{1+9}{2}\cdot 9}{50} = 3 - \frac{9\cancel{0}}{10\cancel{0}} = \frac{21}{10}$$

c) $$\int_0^1 \Phi_n(x)\,dx = \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right)\right] =$$

$$= \frac{1}{n}\left[\left(3-\frac{2}{n}\right) + \left(3-\frac{4}{n}\right) + \dots + \left(3-\frac{2n}{n}\right)\right] =$$

$$= \frac{1}{n}\left[3n - \frac{2(1+2+\dots+n)}{n}\right] = 3 - \frac{\frac{n(n+1)}{\cancel{2}}\cdot\cancel{2}}{n^{\cancel{2}}} =$$

$$= 3 - \frac{n+1}{n} = 3 - 1 - \frac{1}{n} = 2 - \frac{1}{n}$$

$$\int_0^1 \Psi_n(x)\,dx = \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n-1}{n}\right)\right] =$$

$$= \frac{1}{n}\left[(3-0) + \left(3-\frac{2}{n}\right) + \left(3-\frac{4}{n}\right) + \dots + \left(3-\frac{2(n-1)}{n}\right)\right]$$

$$= \frac{1}{n}\left(3n - \frac{2(1+2+\dots+n-1)}{n}\right) = 3 - \frac{n-1}{n} = 2 + \frac{1}{n}$$

d) Vi ser att

$$2-\frac{1}{n} \le \int_0^1 (3-2x)\,dx \le 2+\frac{1}{n} \quad \text{för alla } n.$$

Låt $n \to \infty \Rightarrow \quad 2 \le \int_0^1 (3-2x)\,dx \le 2 \Rightarrow$

$$\underline{\int_0^1 (3-2x)\,dx = 2}.$$

$\boxed{P6}$ $\underline{\underline{11}}$ f är udd $\Leftrightarrow f(-x) = -f(x)$ för alla x.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \left[\begin{array}{l}\text{variabelbyte}\\ x=-t : [-a;0] \to [a;0]\\ dx = -dt \text{ i första}\\ \text{integralen}\end{array}\right]$$

$$= \int_a^0 \underbrace{f(-t)}_{=-f(t)}(-dt) + \int_0^a f(x)\,dx =$$

$$= \int_a^0 f(t)\,dt + \int_0^a f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^a f(x)\,dx =$$

$$= -\cancel{\int_0^a f(x)\,dx} + \cancel{\int_0^a f(x)\,dx} = \underline{0}.$$

$\underline{\underline{13}}$ Se boken, exempel 6.9. Vi har tre spänningar

$u_1 = U\sin\omega t$, $u_2 = U\sin\left(\omega t + \frac{2\pi}{3}\right)$, $u_3 = U\sin\left(\omega t + \frac{4\pi}{3}\right)$

Effektivvärdet för spänningsskillnaden mellan en fas och nollan är 230, vilket betyder att

$$230 = \sqrt{\frac{1}{T}\int_0^T (u_1(t))^2\,dt} =$$

$$= \sqrt{\frac{1}{T}\int_0^T U^2 \sin^2\omega t\,dt} =$$

$$= \sqrt{2\frac{1}{2T} u^2 \int_0^T (1-\cos 2\omega t)\,dt} =$$

$$= \sqrt{\frac{1}{2T} u^2 \left[t - \frac{1}{2\omega}\sin 2\omega t\right]_0^T} = \left[T = \frac{2\pi}{\omega}\right] =$$

$$= \sqrt{\frac{1}{2T} u^2 \left[T - \underbrace{\cancel{\frac{1}{2\omega}\sin 4\pi}}_{=0}\right]} = \frac{u}{\sqrt{2}} \Rightarrow$$

$$u = 230\sqrt{2}.$$

Effektivvärdet av spänningsskillnaden mellan två faser är

$$\sqrt{\frac{1}{T}\int_0^T (u_1(t)-u_2(t))^2\,dt} =$$

$$= \sqrt{\frac{1}{T}\int_0^T u^2\left(\sin\omega t - \sin\left(\omega t + \frac{2\pi}{3}\right)\right)^2 dt} =$$

$$= \sqrt{\frac{1}{T}\int_0^T u^2 \cdot 4\sin^2\frac{\pi}{3}\cos^2\left(\omega t + \frac{\pi}{3}\right) dt} =$$

$$= \sqrt{\frac{1}{T}\cdot 2\sin^2\frac{\pi}{3}\cdot u^2 \int_0^T \left(1+\cos\left(2\omega t + \frac{2\pi}{3}\right)\right) dt}$$

$$= \sqrt{\frac{3}{2T}\cdot(230\sqrt{2})^2\cdot\left(T + \underbrace{\cancel{\left[\sin\left(2\omega t + \frac{2\pi}{3}\right)\right]_0^{T=\frac{2\pi}{\omega}}}}_{=0}\right)} =$$

$$= 230\sqrt{3}.$$