Lektion 23

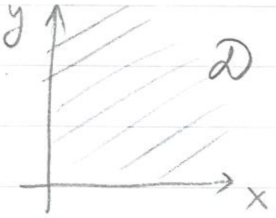
Teori: så länge $f \geq 0$ kan man beräkna $\iint_{\mathcal{D}} f(x,y)\,dx\,dy$ som vanlig genom itererad integration (först x-led, sedan y-led eller tvärtom), även om integralen är generaliserad. Om f byter tecken, skriv

$$\iint_{\mathcal{D}} f(x,y)\,dx\,dy = \underbrace{\iint_{\mathcal{D}^+} \underbrace{f(x,y)}_{>0}\,dx\,dy}_{I_1=} + \underbrace{\iint_{\mathcal{D}^-} \underbrace{f(x,y)}_{<0}\,dx\,dy}_{=I_2}.$$

Int. konvergerar om och endast om både I_1 och I_2 är konvergenta.

6.45

a)



Första kvadranten kan skrivas t ex som

$$\mathcal{D} = \{x \geq 0,\ y \geq 0\}.$$

I så fall

$$\iint_{\mathcal{D}} \underbrace{\frac{xy\,dx\,dy}{(1+x^2+y^2)^3}}_{\geq 0} = \int_0^\infty \left(\int_0^\infty \frac{xy\,dx}{(1+x^2+y^2)^3} \right) dy =$$

$$= \int_0^\infty y \left[-\frac{1}{4}(1+x^2+y^2)^{-2} \right]_{x=0}^{x\to\infty} dy =$$

$$= \int_0^\infty y \left(-\frac{1}{4} \underbrace{\lim_{x\to\infty} \frac{1}{(1+x^2+y^2)^2}}_{=0} + \frac{1}{4}(1+y^2)^{-2} \right) dy$$

$$= \frac{1}{4} \int_0^\infty y(1+y^2)^{-2}\,dy = \frac{1}{4}\left[-\frac{1}{2}(1+y^2)^{-1} \right]_{y=0}^{y\to\infty} =$$

$$= -\frac{1}{8} \lim_{y\to\infty} \frac{1}{1+y^2} + \frac{1}{8} = \boxed{\frac{1}{8}}$$

b) Vi använder polära koordinater:

$$\mathbb{R}^2 = \{0 \leq \varphi \leq 2\pi,\ r \geq 0\}, \text{ så}$$

$$\iint_{\mathbb{R}^2} \underbrace{x^2 e^{-\sqrt{x^2+y^2}}}_{\geq 0}\,dx\,dy = \int_0^\infty \left(\int_0^{2\pi} (r\cos\varphi)^2 e^{-r} \cdot r\,d\varphi \right) dr =$$

1

$$= \int_0^\infty r^3 e^{-r} \left(\underbrace{\int_0^{2\pi} \frac{1+\cos 2\varphi}{2}\, d\varphi}_{=\pi} \right) dr =$$

$$= \pi \int_0^\infty r^3 e^{-r}\, dr = \pi \underbrace{\left[-e^{-r} r^3\right]_{r=0}^{r\to\infty}}_{=0} + \pi \int_0^\infty 3r^2 e^{-r}\, dr =$$

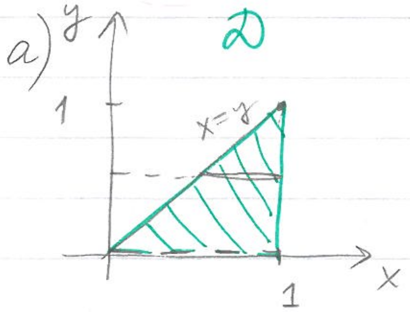
$$= 3\pi \int_0^\infty r^2 e^{-r}\, dr = 3\pi \underbrace{\left[-r^2 e^{-r}\right]_{r=0}^{r\to\infty}}_{=0} + 3\pi \int_0^\infty 2r e^{-r}\, dr$$

$$= 6\pi \int_0^\infty r e^{-r}\, dr = 6\pi \underbrace{\left[-r e^{-r}\right]_{r=0}^{r\to\infty}}_{=0} + 6\pi \int_0^\infty e^{-r}\, dr =$$

$$= 6\pi \left[-e^{-r}\right]_{r=0}^{r\to\infty} = 0 + 6\pi = \boxed{6\pi}$$

6.46

a)



Integralen

$$\iint_{\mathcal{D}} \underbrace{\frac{x}{y}}_{\geq 0}\, dx\,dy$$ är generaliserad längs linjen $y=0$.

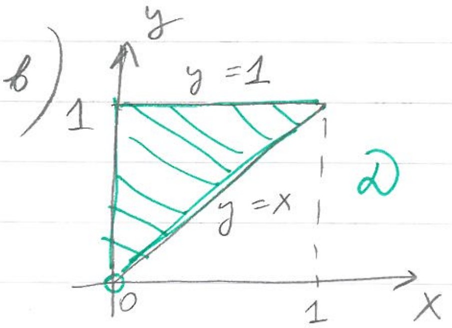
Eftersom $\mathcal{D} = \{0 \leq y \leq 1,\ y \leq x \leq 1\}$ så

$$\iint_{\mathcal{D}} \frac{x}{y}\, dx\,dy = \int_0^1 \left(\int_y^1 \frac{x}{y}\, dx \right) dy =$$

$$= \int_0^1 \frac{1}{y} \left[\frac{x^2}{2} \right]_{x=y}^{x=1} dy = \int_0^1 \left(\frac{1}{2y} - \frac{y}{2} \right) dy$$

$$= \underbrace{\left[\frac{1}{2}\ln|y|\right]_{y\to 0}^{y=1}}_{\text{detta är } \infty} - \left[\frac{y^2}{4}\right]_{y=0}^{y=1} = \infty$$

Integralen är alltså **divergent**

b) 

Integralen

$$\iint_D \underbrace{\frac{x}{y}}_{\geq 0}\,dx\,dy \text{ är generaliserad i } (x,y)=(0,0).$$

Eftersom $D = \{0 \leq x \leq 1,\ x \leq y \leq 1,\ y \neq 0\}$ så

$$\iint_D \frac{x}{y}\,dx\,dy = \int_0^1 \left(\int_x^1 \frac{x}{y}\,dy\right)dx =$$

$$= \int_0^1 \left[x\ln|y|\right]_{y=x}^{y=1} dx =$$

$$= \int_0^1 \left(0 - x\ln|x|\right) dx =$$

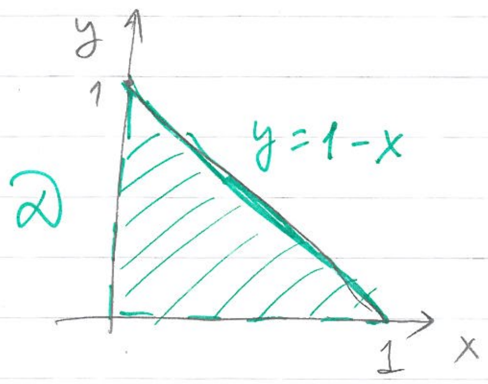
$$= -\int_0^1 x\ln x\,dx = \qquad = \left[\frac{x^2}{4}\right]_{x=0}^{x=1}$$

$$= -\left[\frac{x^2}{2}\ln x\right]_{x\to 0}^{x=1} + \int_0^1 \frac{x}{2}\,dx =$$

$$= -0 + \lim_{x\to 0}\frac{x^2\ln x}{2} + \frac{1}{4} = \left[\begin{array}{l} t = \frac{1}{x} \\ x\to 0 \Leftrightarrow t\to\infty \end{array}\right]$$

$$= \frac{1}{4} + \lim_{t\to\infty}\frac{\ln\frac{1}{t}}{2t^2} = \frac{1}{4} - \underbrace{\lim_{t\to\infty}\frac{\ln t}{2t^2}}_{\to 0 \text{ enligt hastighetstabellen}} = \boxed{\frac{1}{4}}$$

6.47



Integralen $\iint_{\mathcal{D}} \frac{dx\,dy}{\sqrt{xy}}$ är generaliserad längs linjer $x=0$ och $y=0$.

$\sqrt{xy} \geq 0$

Eftersom

$\mathcal{D} = \{ 0 < x \leq 1,\ 0 < y < 1-x \}$ så

$$\iint_{\mathcal{D}} \frac{dx\,dy}{\sqrt{xy}} = \int_0^1 \left(\int_0^{1-x} x^{-\frac{1}{2}} y^{-\frac{1}{2}}\, dy \right) dx =$$

$$= \int_0^1 x^{-\frac{1}{2}} \left[\frac{y^{\frac{1}{2}}}{\frac{1}{2}} \right]_{y \to 0}^{y=1-x} dx =$$

$$= \int_0^1 2x^{-\frac{1}{2}} (1-x)^{\frac{1}{2}}\, dx = 2\int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}}\, dx =$$

$$= \left[\begin{array}{ll} \sqrt{x} = t & 0 \leq x \leq 1 \Rightarrow \\ dt = \frac{dx}{2\sqrt{x}} & 0 \leq t \leq 1 \\ & x = t^2 \end{array} \right] =$$

$$= 4\int_0^1 \sqrt{1-t^2}\, dt = \otimes$$

Observera att $\int_0^1 \sqrt{1-t^2}\, dt = \int_0^1 1 \cdot \sqrt{1-t^2}\, dt =$

$$= \cancel{\left[t\sqrt{1-t^2} \right]_{t=0}^{t=1}} + \int_0^1 \frac{t^2}{\sqrt{1-t^2}}\, dt = -\int_0^1 \frac{(1-t^2)-1}{\sqrt{1-t^2}}\, dt =$$

$$= -\int_0^1 \sqrt{1-t^2}\, dt + \underbrace{\int_0^1 \frac{dt}{\sqrt{1-t^2}}}_{[\arcsin t]_{t=0}^{t=1}} = -\int_0^1 \sqrt{1-t^2}\, dt + \frac{\pi}{2} \Rightarrow$$

4

$$2\int_0^1 \sqrt{1-t^2}\,dt = \frac{\pi}{2} \Rightarrow \int_0^1 \sqrt{1-t^2}\,dt = \frac{\pi}{4}, \text{ så}$$

$$\otimes = \frac{\pi}{4} \cdot 4 = \boxed{\pi}$$

6.51 Vi byter mot sfäriska koordinater

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{matrix} r \geq 0 \\ 0 \leq \theta \leq \pi \\ 0 \leq \varphi \leq 2\pi \end{matrix} \qquad \text{så}$$

$$\iiint_{\mathcal{D}} \overset{\geq 0}{\frac{\exp\left(-\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}}}\,dx\,dy\,dz =$$

$$= \int_0^\infty \left(\int_0^\pi \left(\int_0^{2\pi} \frac{e^{-r}}{r} r^2 \sin\theta \, d\varphi \right) d\theta \right) dr$$

$$= \int_0^\infty \left(\int_0^\pi 2\pi e^{-r} \cdot r \cdot \sin\theta \, d\theta \right) dr$$

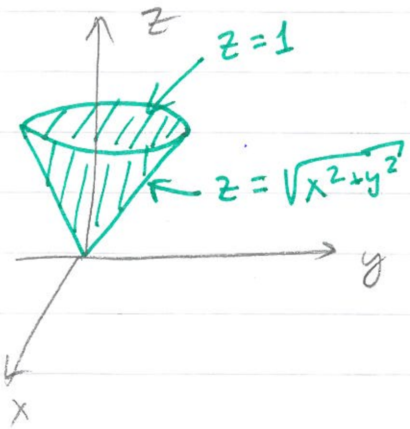
$$= \int_0^\infty 2\pi r e^{-r} \underbrace{\Big[-\cos\theta \Big]_{\theta=0}^{\theta=\pi}}_{=2} dr =$$

$$= 4\pi \int_0^\infty \overset{\downarrow}{r} \overset{\uparrow}{e^{-r}} dr = 4\pi \left(\Big[-re^{-r} \Big]_{r=0}^{r\to\infty} + \right.$$

$$\left. + \int_0^\infty e^{-r} dr \right) = 4\pi \left(\lim_{r\to\infty} \left(-\frac{r}{e^r} \right) + 0 + \right.$$

$$\left. + \Big[-e^{-r} \Big]_{r=0}^{r\to\infty} \right) = \boxed{4\pi}$$

6.52



Vi projicerar D på z-axeln $\Rightarrow$ $\boxed{0 \le z \le 1}$

När $z = \text{konst}$ så

$\sqrt{x^2+y^2} < z \Rightarrow$

skivorna är $\boxed{D_z = \{\sqrt{x^2+y^2} < z\}}$.

$$\iiint_D \frac{dx\,dy\,dz}{\underbrace{\sqrt{x^2+y^2+z^2}}_{\geq 0}} = \int_0^1 \left(\iint_{D_z} \frac{dx\,dy}{\sqrt{x^2+y^2+z^2}} \right) dz =$$

$$= \left[\begin{array}{l} D_z = \{\sqrt{x^2+y^2} < z\} \text{ då } z = \text{konst} \Rightarrow \text{byter} \\ \text{mot polära koordinater} \\ \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow 0 \le r \le z,\ 0 \le \varphi \le 2\pi \end{array} \right] =$$

$$= \int_0^1 \left(\int_0^{2\pi} \left(\int_0^z \frac{r\,dr}{\sqrt{r^2+z^2}} \right) d\varphi \right) dz =$$

$$= \int_0^1 \left(\int_0^{2\pi} \left(\left[\sqrt{r^2+z^2} \right]_{r=0}^{r=z} \right) d\varphi \right) dz =$$

$$= \int_0^1 \left(\int_0^{2\pi} (z\sqrt{2} - z)\, d\varphi \right) dz =$$

$$= \int_0^1 z(\sqrt{2}-1) \cdot 2\pi\, dz = 2\pi(\sqrt{2}-1) \left[\frac{z^2}{2} \right]_{z=0}^{z=1} =$$

$$= \boxed{\pi(\sqrt{2}-1)}.$$

6.53

a) Problemet är att $\frac{x}{x^2+y^2+1} = \begin{cases} \geq 0 & \text{då } x \geq 0 \\ \leq 0 & \text{då } x \leq 0 \end{cases}$

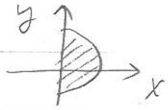
så itererad integration kan inte tillämpas direkt. Vi måste skriva

$$I = \iint_{\mathbb{R}^2} \frac{x\,dx\,dy}{\sqrt{x^2+y^2+1}} = \iint_{x\geq 0} \frac{x\,dx\,dy}{\sqrt{x^2+y^2+1}} + \iint_{x \leq 0} \frac{x\,dx\,dy}{\sqrt{x^2+y^2+1}}$$

$$= \iint_{x\geq 0} \underbrace{\frac{x\,dx\,dy}{\sqrt{x^2+y^2+1}}}_{\geq 0} - \iint_{x\leq 0} \underbrace{\frac{(-x)\,dx\,dy}{\sqrt{x^2+y^2+1}}}_{\geq 0}$$

$$= I_1 - I_2 .$$

Integralen blir konvergent om och endast om både I_1 och I_2 är konvergenta. Men



$$I_1 = \left[\begin{array}{l}\text{polära,} \\ 0 \leq r < \infty,\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\end{array}\right] = \int_0^\infty \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2 \cos\varphi}{\sqrt{r^2+1}}\, d\varphi \right) dr =$$

$$= \int_0^\infty \left(\frac{r^2}{\sqrt{r^2+1}} \Big[+\sin\varphi \Big]_{\varphi=-\frac{\pi}{2}}^{\varphi=\frac{\pi}{2}} \right) dr =$$

alternativ: $I_1 = \int_{-\infty}^{\infty} \Big(\underbrace{\int_0^\infty \frac{x\,dx}{\sqrt{\dots}}}_{\text{divergent}} \Big) dy$

$$= \int_0^\infty \frac{2r^2}{\sqrt{r^2+1}}\, dr \quad \text{är divergent då}$$

$\left.\begin{array}{l} * \int_0^\infty r^2\,dr \text{ är divergent} \\ * \frac{2}{\sqrt{r^2+1}} \leq 2 \text{ är en positiv begr. funktion} \end{array}\right\}$ 2a jämf. satsen

$\Rightarrow$ I är också divergent.

b) Samma problem!

$$\frac{x}{(x^2+y^2)^2+1} = \begin{cases} \geq 0 & \text{då } x \geq 0 \\ \leq 0 & \text{då } x \leq 0 \end{cases} \Rightarrow$$

vi undersöker

$$I = \iint_{\mathbb{R}^2} \frac{x}{(x^2+y^2)^2+1}\,dx\,dy = \iint_{x\geq 0} \frac{x\,dx\,dy}{(x^2+y^2)^2+1} + \iint_{x \leq 0} \frac{x\,dx\,dy}{(x^2+y^2)^2+1}$$

$$= \iint_{x\geq 0} \underbrace{\frac{x\,dx\,dy}{(x^2+y^2)^2+1}}_{\geq 0} - \iint_{x\leq 0} \underbrace{\frac{(-x)\,dx\,dy}{(x^2+y^2)^2+1}}_{\geq 0}$$

$$= I_1 + I_2.$$

$$I_1 = \left[\begin{array}{l}\text{polära} \\ 0 \leq r < \infty,\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\end{array}\right] = \int_0^\infty \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2 \cos\varphi\, d\varphi}{r^4+1} \right) dr =$$

$$= 2\int_0^\infty \frac{r^2\,dr}{r^4+1} \text{ är konvergent då}$$

$$\int_0^\infty \frac{r^2}{r^4+1}\,dr = \underbrace{\int_0^1 \frac{r^2\,dr}{r^4+1}}_{\text{en vanlig integral, har ett ändligt värde}} + \underbrace{\int_1^\infty \frac{r^2\,dr}{r^4+1}}_{\leq \int_1^\infty \frac{r^2 dr}{r^4} = \int_1^\infty \frac{dr}{r^2} \text{ - konvergent}}.$$

$$I_2 = \int_0^\infty \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{-r^2 \cos\varphi}{r^4+1}\,d\varphi \right) dr = -2\int_0^\infty \frac{r^2\,dr}{r^4+1}$$

är också konvergent $\Rightarrow$

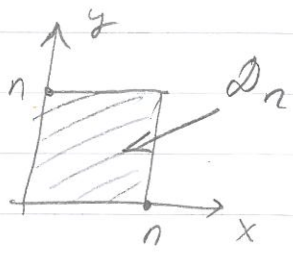
I är konvergent med värdet

$$I = I_1 + I_2 = 2\int_0^\infty \frac{r^2\,dr}{r^4+1} - 2\int_0^\infty \frac{r^2\,dr}{r^4+1} = \boxed{0.}$$

Extra

6.44

a) $\iint_{\mathcal{D}_n} e^{-x-y}\,dx\,dy =$

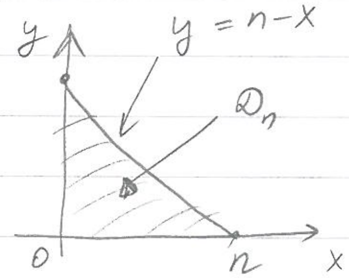


$= \int_0^n \left(\int_0^n e^{-x-y}\,dx \right) dy =$

$= \int_0^n e^{-x}\,dx \cdot \int_0^n e^{-y}\,dy = \left(\left[-e^{-x} \right]_{x=0}^{x=n} \right)^2 =$

$= (1-e^{-n})^2 = \boxed{1-2e^{-n}+e^{-2n}}$

b) $\iint_{\mathcal{D}_n} e^{-x-y}\,dx\,dy =$



$0 \le x \le n$
$0 \le y \le n-x$

$= \int_0^n \left(\int_0^{n-x} e^{-x} \cdot e^{-y}\,dy \right) dx =$

$= \int_0^n e^{-x} \left[-e^{-y} \right]_{y=0}^{y=n-x} dx =$

$= \int_0^n e^{-x} \left(-e^{-n+x} + 1 \right) dx =$

$= \int_0^n \left(-e^{-n} + e^{-x} \right) dx = -ne^{-n} - \left[e^{-x} \right]_{x=0}^{x=n} =$

$= -ne^{-n} - e^{-n} + 1 = \boxed{1-(n+1)e^{-n}}$

c) När $n \to \infty$, $\mathcal{D}_n \to \mathcal{D}$, där $\mathcal{D}$ är första kvadrant, så

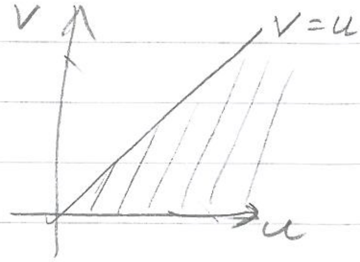
$\iint_{\mathcal{D}} e^{-x-y}\,dx\,dy = \lim_{n\to\infty} \iint_{\mathcal{D}_n} e^{-x-y}\,dx\,dy \overset{a)}{=} \lim_{n\to\infty} (1-e^{-n})^2 = \boxed{1}$

6.48 Vi gör ett variabelbyte

$$\begin{cases} x+y=u \\ y=v \end{cases} \Leftrightarrow \begin{cases} x=u-v \\ y=v \end{cases} \qquad \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\mathcal{D} = \{x \geq 0,\ y \geq 0\} = \{u-v \geq 0,\ v \geq 0\} = \{u \geq v \geq 0\}$$

Integranden beror bara på $u \Rightarrow$ det är bäst att integrera i v-led först:



$$\iint_{\mathcal{D}} \frac{dx\,dy}{\underbrace{1+(x+y)^4}_{\geq 0}} = \int_0^{\infty} \left(\int_0^{u} \frac{dv}{1+u^4} \right) du = \qquad \begin{matrix} 0 \leq u < \infty \\ 0 \leq v \leq u \end{matrix}$$

$$= \int_0^{\infty} \frac{u}{1+u^4}\, du = \frac{1}{2} \left[\arctan u^2 \right]_{u=0}^{u \to \infty} =$$

$$= \boxed{\frac{\pi}{4}}$$

6.49 Eftersom $-x^2 - xy - y^2 = -\left(x + \frac{1}{2}y\right)^2 - \left(\frac{\sqrt{3}}{2}y\right)^2$, gör variabelbyte

$$\begin{cases} u = x + \frac{1}{2}y \\ v = \frac{\sqrt{3}}{2}y \end{cases}, \qquad \frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{vmatrix} = \frac{\sqrt{3}}{2}$$

Integrationsområdet $(x,y) \in \mathbb{R}^2$ blir $(u,v) \in \mathbb{R}^2$.

$$\iint_{\mathbb{R}^2} e^{-x^2-xy-y^2} \, dx\,dy = \iint_{\mathbb{R}^2} e^{-u^2-v^2} \cdot \frac{2}{\sqrt{3}} \, du\,dv = \left[\begin{matrix}\text{polära} \\ \text{koord.}\end{matrix}\right]$$

≥ 0

$$= \frac{2}{\sqrt{3}} \int_0^{2\pi} \left(\int_0^{\infty} e^{-r^2} \cdot r \, dr \right) d\varphi = \frac{4\pi}{\sqrt{3}} \left[-\frac{e^{-r^2}}{2} \right]_{r=0}^{r \to \infty} =$$

$$= \frac{4\pi}{\sqrt{3}} \left(0 + \frac{1}{2} \right) = \boxed{\frac{2\pi}{\sqrt{3}}}$$