1. Vektorkalkyl

Uppgift 1.1 (Sid. 1)

Lösning

$$\left.\begin{array}{l} (1)\ a\cdot b=(1,2,2)\cdot(2,-2,1)=2-4+2=0 \Leftrightarrow a\perp b \\ (2)\ a\cdot c=(1,2,2)\cdot(2,1,-2)=2+2-4=0 \Leftrightarrow a\perp c \\ (3)\ b\cdot c=(2,-2,1)\cdot(2,1,-2)=4-2-2=0 \Leftrightarrow b\perp c \end{array}\right\} \Rightarrow$$

$\Rightarrow \underline{a\perp b\perp c\perp a}$, V.S.V.

Uppgift 1.2 (Sid. 1)

Lösning

(1) $a\cdot b=(1,1,0)\cdot(1,0,1)=1+0+0=1$

(2) $|a|=\sqrt{1^2+1^2+0^2}=\sqrt{2}$; $\quad |b|=\sqrt{1^2+0^2+1^2}=\sqrt{2}$;

(3) $a\cdot b=|a|\cdot|b|\cos\theta \Leftrightarrow 1=2\cos\theta \Leftrightarrow \cos\theta=\frac{1}{2} \Rightarrow \underline{\theta=\frac{\pi}{3}}$.

Övning 1.3 (Sid. 1)

Lösning:

$\underline{u=(1,2,a)}$, $\underline{v=(1,1,0)}$, $\underline{w=(0,1,1)}$.

$$\left.\begin{array}{l} u\cdot v=|u|\cdot|v|\cos\theta \\ u\cdot v=|u|\cdot|w|\cos\theta \end{array}\right\} \Rightarrow \frac{u\cdot v}{u\cdot w}=\frac{|u|\cdot|v|\cos\theta}{|u|\cdot|w|\cos\theta}=\frac{|v|}{|w|} \Leftrightarrow$$

$$\Leftrightarrow \frac{(1,2,a)\cdot(1,1,0)}{(1,2,a)\cdot(0,1,1)}=1 \Leftrightarrow \frac{3}{2+a}=1 \Leftrightarrow \underline{a=1}.$$

Uppgift 1.3 (Sid. 1)

Lösning

Antag att $u=(a,b,c)$ är normerad till 1 och att u är ortogonal mot såväl $v=(-1,0,1)$ som $w=(1,1,1)$.

$$\left.\begin{array}{l} (1)\ u\perp v \Rightarrow u\cdot v=(a,b,c)\cdot(-1,0,1)=-a+c=0 \\ (2)\ u\perp w \Rightarrow u\cdot w=(a,b,c)\cdot(1,1,1)=a+b+c=0 \\ (3)\ |u|=1 \Rightarrow a^2+b^2+c^2=1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} a=c \\ b=-2c \\ a^2+b^2+c^2=1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a=c \\ b=-2c \\ 6c^2=1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a=c \\ b=-2c \\ c^2=1/6 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a=c \\ b=-2c \\ c=\pm 1/\sqrt{6} \end{array}\right.$$

$$\Leftrightarrow \underline{u_1=\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)} \vee \underline{u_2=\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)}.$$

Anm. $u_1=-u_2$, dvs $u_1 \uparrow\downarrow u_2$.

Uppgift 1.5 (Sid. 1)

Lösning

$u=(1,1,1)$, $v=(1,0,a)$, $w=(3,4,-7)$

(1) $u \perp u\times v \perp v$ (def. på kryssprodukten).

(2) $w \parallel u\times v \Rightarrow v \perp w \Leftrightarrow v\cdot w = (1,0,a)\cdot(3,4,-7) =$

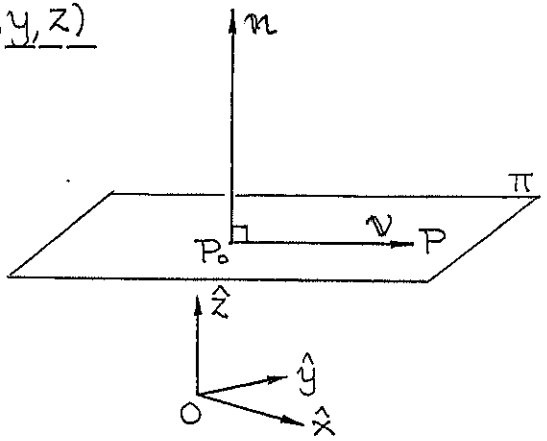
$= 3-7a = 0 \Leftrightarrow 7a = 3 \Leftrightarrow \underline{a=\frac{3}{7}}$.

Uppgift 1.6 (Sid. 1)

Lösning

$P_0:(1,2,3)$; $n=(4,5,6)$.

$P:(x,y,z)$



$n\perp v \Leftrightarrow n\cdot v = 0 \Leftrightarrow n\cdot\overrightarrow{P_0P} = n\cdot(\overrightarrow{OP}-\overrightarrow{OP_0}) = 0 \Leftrightarrow$

$\Leftrightarrow (4,5,6)\cdot(x-1,\,y-2,\,z-3) = 4(x-1)+5(y-2)+6(z-3) = 0$

$\Leftrightarrow 4x+5y+6z-4-10-18=0 \Leftrightarrow 4x+5y+6z-32=0.$

Svar: $\pi: 4x+5y+6z=32.$

Uppgift 1.6 (Sid. 1)

Lösning

$P_1:(1,2,3)$, $P_2:(3,3,5)$, $P_3:(3,1,6)$

Planets ekvation är av formen

$$\pi: Ax+By+Cz=D.$$

$$\begin{cases} P_1\in\pi \Rightarrow A+2B+3C=D \\ P_2\in\pi \Rightarrow 3A+3B+5C=D \\ P_3\in\pi \Rightarrow 3A+B+6C=D \end{cases} \overset{(-3)}{\Leftrightarrow} \begin{cases} A+2B+3C=D \\ -3B-4C=-2D \\ -5B-3C=-2D \end{cases}$$

$$\Leftrightarrow \begin{cases} A+2B+3C=D \\ 3B+4C=2D \\ 5B+3C=2D \end{cases} \overset{(-2)}{\Leftrightarrow} \begin{cases} A+2B+3C=D \\ 3B+4C=2D \\ -B-5C=-2D \end{cases} \overset{(3)\,(2)}{\Leftrightarrow}$$

$$\Leftrightarrow \begin{cases} A \quad -7C=-3D \\ -11C=-4D \\ B+5C=2D \end{cases} \overset{(-1/11)}{\Leftrightarrow} \begin{cases} A \quad -7C=-3D \\ C=4D/11 \\ B+5C=2D \end{cases} \overset{(7)\,(-5)}{\Leftrightarrow}$$

$$\Leftrightarrow \begin{cases} A = -5D/11 \\ B = 2D/11 \\ C = 4D/11 \end{cases} \Rightarrow -\frac{5}{11}Dx + \frac{2}{11}Dy + \frac{4}{11}Dz = D \Rightarrow$$

$\Rightarrow / D = -11 / \Rightarrow$ $\underline{5x - 2y - 4z = -11}$.

Svar: $\pi: 5x - 2y - 4z + 11 = 0$.

Annan lösning

$$\left.\begin{aligned} u &= \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (3,3,5) - (1,2,3) = (2,1,2) \\ v &= \overrightarrow{P_1P_3} = \overrightarrow{OP_3} - \overrightarrow{OP_1} = (3,1,6) - (1,2,3) = (2,-1,3) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow n = u \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = 5\hat{x} - 2\hat{y} - 4\hat{z} = (5,-2,-4) \Rightarrow$$

$\pi: 5x - 2y - 4z = k$; $P_1 \in \pi \Rightarrow k = 5 \cdot 1 - 2 \cdot 2 - 4 \cdot 3 = -11$.

Anm. $Ax + By + Cz = D$ är ekvationen för ett plan med normalvektorn $n = (A,B,C)$.

Det mesta under rubriken "vektorkalkyl" genomgås i den linjära algebran; vektorerna skall i denna kurs vara radvektorer.

2. Ytor, normaler

Uppgift 2.1 (Sid. 1)

Lösning

$S: 4x^2 + 4y^2 + z^2 = 4$; $P_0: (\frac{1}{2}, \frac{1}{2}, \sqrt{2})$.

$f(x,y,z) = 4x^2 + 4y^2 + z^2 \Rightarrow f(x,y,z) = f(P_0) \Rightarrow S$ är en nivåyta till f (som omfattar P_0).

$\operatorname{grad} f(x,y,z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (8x, 8y, 2z)$;

$n = \operatorname{grad} f(P_0) = (4, 4, 2\sqrt{2}) = 2(2,2,\sqrt{2}) = 2 \cdot v$;

Tangentplanets ekvation blir alltså

$(2,2,\sqrt{2}) \cdot (x - \frac{1}{2}, y - \frac{1}{2}, z - \sqrt{2}) = 0 \Leftrightarrow \underline{2x + 2y + \sqrt{2}z = 4}$.

Anm. Låt S vara ytan som består av punkterna (x,y,z) s.a. $f(x,y,z) = k$, där k är en konstant. S:s tangentplan i pkt $P_0: (x_0, y_0, z_0)$ definieras av

$$\operatorname{grad} f(P_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

om $\operatorname{grad} f(P_0) \neq 0$. Tangentplanet är alltså

$\pi = \{(x,y,z) : \text{grad}f(P_0)\cdot(x-x_0, y-y_0, z-z_0)=0\}$.

Uppgift 2.2 (Sid. 1)

Lösning

$f(x,y,z) = x^2+y^2+z^2 \Rightarrow \text{grad}f(x,y,z) = 2(x,y,z) \Rightarrow$

$n = \text{grad}f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) = 2\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}(1,1,1) \Rightarrow$

$\hat{n} = \frac{n}{|n|} = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Uppgift 2.3 (Sid. 1)

Lösning

$f(x,y,z) = x^2+2xy+2xz-2x+2y+2z-2$; $f(0,1,0)=0$.

$g(x,y,z) = x-y+z+1$; $g(0,1,0)=0$.

Vinkeln mellan ytorna $S_1: f(x,y,z)=0$ och $S_2: g(x,y,z)=0$ i punkten $P_0:(0,1,0)$ är lika med vinkeln mellan motsvarande tangentplan π_1 resp. π_2 dvs vinkeln mellan dessas normaler; normalerna är parallella med gradienterna $\nabla f(P_0)$ resp. $\nabla g(P_0)$ så det gäller att bestämma vinkeln mellan $\nabla f(P_0)$ och $\nabla g(P_0)$.

$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x+2y+2z-2, 2x+2, 2x+2)$

$\left.\begin{array}{l}\nabla f(0,1,0) = (0,2,2)\\ \nabla g(0,1,0) = (1,-1,1)\end{array}\right\} \Rightarrow / \nabla f(P_1)\cdot\nabla g(P_1) = 0 / \Rightarrow S_1 \perp S_2$.

Svar: Vinkeln $\frac{\pi}{2}$.

Uppgift 2.4 (Sid. 1)

Lösning

a) $x^2-2y^2-2z=0 \Leftrightarrow 2z = x^2-2y^2 \Leftrightarrow z = \frac{x^2}{2}-y^2 \Rightarrow$

$$\Rightarrow \begin{cases} x = 2u \\ y = v \\ z = 2u^2-v^2 \end{cases} \Rightarrow \gamma: r(u,v) = (2u, v, 2u^2-v^2).$$

b) $x^2-2y^2=6 \Leftrightarrow \frac{x^2}{6}-\frac{y^2}{3}=1 \Rightarrow \begin{cases} x = \sqrt{6}\cosh u \\ y = \sqrt{3}\sinh u \\ z = v \end{cases} \Rightarrow$

$\Rightarrow \gamma: r(u,v) = (\sqrt{6}\cosh u, \sqrt{3}\sinh u, v)$.

Uppgift 2.5 (Sid. 1)

Lösning

a) $\gamma: \mathbf{r}(u,v) = (2u, v, 2u^2 - v^2)$;

$\mathbf{r}(u_0, v_0) = (2,1,1) \Leftrightarrow (u_0, v_0) = (1,1)$;

$\left.\begin{aligned} \mathbf{r}'_u &= (2, 0, 4u) \Rightarrow \mathbf{r}'_u(1,1) = (2,0,4) \\ \mathbf{r}'_v &= (0, 1, -2v) \Rightarrow \mathbf{r}'_v(1,1) = (0,1,-2) \end{aligned}\right\} \Rightarrow \mathbf{n} = \mathbf{r}'_u \times \mathbf{r}'_v =$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 0 & 4 \\ 0 & 1 & -2 \end{vmatrix} = (-4, 4, 2).$$

b) $F(x,y,z) = x^2 - 2y^2 - 2z \Rightarrow \operatorname{grad} F(x,y,z) = (2x, -4y, -2)$

$\Rightarrow \mathbf{n} = \operatorname{grad} F(2,1,1) = (4, -4, -2)$.

Svar: a) $\mathbf{n} = (-2, 2, 1)$; b) $\mathbf{n} = (2, -2, -1)$.

Uppgift 2.6 (Sid. 1)

Lösning

$\gamma: \mathbf{r}(t,u) = (e^t, e^{-t}, u)$, $0 \le t, u < \infty$.

(1) $\mathbf{r}(t_0, u_0) = (1,1,1) \Leftrightarrow (e^{t_0}, e^{-t_0}, u_0) = (1,1,1) \Leftrightarrow$

$\Leftrightarrow t_0 = 0 \wedge u = 1$. forts

(2) $\left.\begin{aligned} \mathbf{r}'_t &= (e^t, -e^{-t}, 0) \Rightarrow \mathbf{r}'_t(0,1) = (1,-1,0) \\ \mathbf{r}'_u &= (0,0,1) = \mathbf{r}'_u(0,1) \end{aligned}\right\} \Rightarrow \mathbf{n} = \mathbf{r}'_t \times \mathbf{r}'_u =$

$= (\hat{x} - \hat{y}) \times \hat{z} = (-1, -1, 0) \Rightarrow \mathbf{n} = (1,1,0)$ normalvektor i punkten $(1,1,1)$.

(3) Tangentplanets ekvation blir således

$(1,1,0) \cdot (x-1, y-1, z-1) = 0 \Leftrightarrow x + y = 2$.

Uppgift 2.8 (Sid. 2)

Lösning

$f(x,y,z) = x^2 + y^2 \Rightarrow \operatorname{grad} f(x,y,z) = (2x, 2y, 0) \Rightarrow$

$\Rightarrow |\operatorname{grad} f(x,y,z)| = 2\sqrt{x^2+y^2} = 2 \cdot 2 = 4$, ty $x^2 + y^2 = 4$;

$\hat{n} = -\frac{\nabla f}{|\nabla f|} = -\frac{1}{4}(2x, 2y, 0) = \left(-\frac{x}{2}, -\frac{y}{2}, 0\right)$.

Uppgift 2.9 (Sid. 2)

Lösning

$S: f(x,y) = 2x^3 - x^2 y - y^2 - 2y + 11$; $P_0: (2, 3, 0)$.

Vinkeln mellan tangentplanet π till S i P_0

och xy-planet är lika med vinkeln mellan deras normaler.

$F(x,y,z) = z - f(x,y) = z - 2x^3 + x^2y + y^2 + 2y - 11$;

$\nabla F(x,y,z) = (-6x^2 + 2xy,\ x^2 + 2y + 2,\ 1) \Rightarrow \underline{\nabla F(P_0) = (-12, 12, 1)}$.

Vinkeln (det är den spetsiga som avses) är θ.

$\nabla F(P_0) \cdot \hat{z} = |\nabla F(P_0)| \cdot 1 \cdot \cos\theta \Rightarrow 1 = 17\cos\theta \Leftrightarrow \cos\theta = \frac{1}{17}$

Svar: Vinkeln i fråga är $\arccos\frac{1}{17} \approx 1{,}51$ rad $\approx 86{,}6°$.

Övning 2.10 (Sid. 2)

Lösning: $P_0 : (x_0, y_0, z_0)$.

$f(x,y,z) = x^3 + y^3 + 2z^3 \Rightarrow \operatorname{grad} f(x,y,z) = (3x^2, 3y^2, 6z^2)$

$\Rightarrow \operatorname{grad} f(P_0) = 3(x_0^2, y_0^2, 2z_0^2) = 3 \cdot n$, n normal.

Tangentplanets ekvation är

$$\pi : (x_0^2, y_0^2, 2z_0^2) \cdot (x - x_0, y - y_0, z - z_0) = 0 \Leftrightarrow$$

$$\Leftrightarrow \pi : x_0^2 x + y_0^2 y + 2z_0^2 z = x_0^3 + y_0^3 + 2z_0^3 = 1.$$

$P_1 : (2,2,1) \in \pi \Rightarrow 2x_0^2 + 2y_0^2 + 2z_0^2 = 1 \Leftrightarrow x_0^2 + y_0^2 + z_0^2 = \frac{1}{2}$;

Svar: Det sökta avståndet är $1/\sqrt{2}$.

Uppgift 2.11 (Sid. 2)

Lösning $\qquad P_0 : (1,-2,1)$

$S_1 : f(x,y,z) = xy^2z - 3x - z^2$; $S_2 : g(x,y,z) = 3x^2 - y^2 + 2z$

$$\left.\begin{array}{l} f(1,-2,1) = 0 \Rightarrow P_0 \in S_1 \\ g(1,-2,1) = 1 \Rightarrow P_0 \in S_2 \end{array}\right\} \Rightarrow P_0 \in S_1 \cap S_2$$

Vinkeln mellan S_1 och S_2 i den gemensamma punkten P_0 är lika med vinkeln mellan deras tangentplan i P_0, dvs vinkeln mellan deras normaler (gradienter):

$\nabla f(x,y,z) = (y^2z - 3,\ 2xyz,\ xy^2 - 2z) \Rightarrow \nabla f(P_0) = (1,-4,2)$;

$\nabla g(x,y,z) = (6x, -2y, 2) \Rightarrow \nabla g(P_0) = (6, 4, 2) = 2(3, 2, 1)$;

$(1,-4,2) \cdot (3, 2, 1) = \sqrt{21} \cdot \sqrt{14} \cos\theta \Leftrightarrow -3 = 7\sqrt{6}\cos\theta \Rightarrow$

$\Rightarrow \cos\theta = -\frac{3}{7\sqrt{6}} = -\frac{\sqrt{6}}{14} \Leftrightarrow \theta = \pi - \arccos\frac{\sqrt{6}}{14}$.

Svar: $\arccos\frac{\sqrt{6}}{14}$. (Det är den spetsiga som avses.)

3. # Areor

Arean $\mathcal{A}(S)$ av en parametriserad yta definieras av

$$\mathcal{A}(D)=\iint_D |r'_u \times r'_v|\,du\,dv$$

där $|r'_u \times r_v|$ är "normen" av $r'_u \times r'_v$. Om S är unionen av S_i, blir $\mathcal{A}(S)$ summan av delareorna, $\sum_i \mathcal{A}(S_i)$.

Uppgift 3.1 (Sid. 2)

Lösning

$$z=f(x,y):=\frac{2}{3}(x^{3/2}+y^{3/2}),\quad D=[0,1].$$

$$\left.\begin{matrix} x=u^2 \\ y=v^2 \end{matrix}\right\} \Rightarrow z=f(u^2,v^2)=\frac{2}{3}(u^3+v^3),\ 0\le u,v\le 1;$$

$$S:\ r(u,v)=(u^2,v^2,\tfrac{1}{2}(u^3+v^3)),\quad D=[0,1]\times[0,1].$$

$$r'_u \times r_v=\hat{x}\begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix}+\hat{y}\begin{vmatrix}\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\end{vmatrix}+\hat{z}\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=$$

$$=\begin{vmatrix}0 & 2v\\ 2u^2 & 2v^2\end{vmatrix}\hat{x}+\begin{vmatrix}2u^2 & 2v^2\\ 2u & 0\end{vmatrix}\hat{y}+\begin{vmatrix}2u & 0\\ 0 & 2v\end{vmatrix}\hat{z}=$$

$$=-4u^2v\,\hat{x}-4uv^2\hat{y}+4uv\,\hat{z}=-4uv(u,v,-1)\Rightarrow$$

$$\Rightarrow |r'_u\times r'_v|=|-4uv(u,v,-1)|=4uv\sqrt{u^2+v^2+1}\Rightarrow$$

$$\mathcal{A}(S)=\iint_D 4uv\sqrt{u^2+v^2+1}\,du\,dv=$$

$$=\int_0^1\Big(\int_0^1\sqrt{u^2+v^2+1}\,2u\,du\Big)2v\,dv=$$

$$=\int_0^1\Big(\Big[\frac{2}{3}(u^2+v^2+1)^{3/2}\Big]_{u=0}^{u=1}\Big)2v\,dv=$$

$$=\frac{2}{3}\int_0^1\big((v^2+2)^{3/2}-(v^2+1)^{3/2}\big)2v\,dv=$$

$$=\frac{2}{3}\cdot\Big[\frac{2}{5}\big((u^2+2)^{5/2}-(u^2+1)^{5/2}\big)\Big]_0^1=$$

$$=\frac{4}{15}(3^{5/2}-2^{5/2}-2^{5/2}+1)=\frac{4}{15}(3^{5/2}-2\cdot 2^{5/2}-1)=$$

$$=\frac{4}{15}(9\sqrt{3}-8\sqrt{2}+1).$$

Uppgift 2.2 (Sid. 2)

Lösning

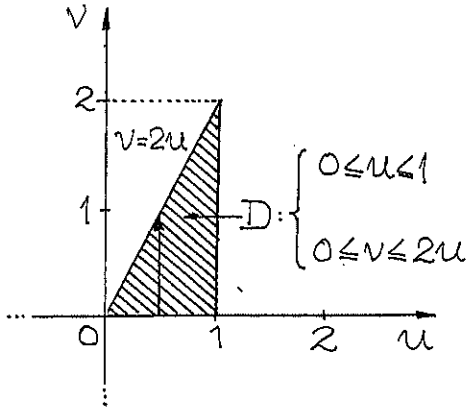
$$S:\ z=f(x,y)=\frac{1}{2}x^2+y;\quad D:\begin{cases}0\le x\le 1\\ 0\le y\le 2x\end{cases};$$

$$S:\ r(u,v)=(u,v,\tfrac{1}{2}u^2+v),\ (u,v)\in D;$$

Man kan behålla x och y som parametrar.

$$r'_u \times r'_v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & u \\ 0 & 1 & 1 \end{vmatrix} = -u\hat{x} - \hat{y} + \hat{z} = (-u, -1, 1) \Rightarrow$$

$$\Rightarrow d\sigma = |r'_u \times r'_v| \, du dv = \underline{\sqrt{u^2+2}\, du dv};$$



$$\mathcal{A}(S) = \iint_D d\sigma = \int_0^1 du \int_0^{2u} \sqrt{u^2+2}\, dv =$$

$$= \int_0^1 \sqrt{u^2+2} \cdot 2u\, du =$$

$$= \left[\frac{2}{3}(u^2+2)^{3/2}\right]_0^1 =$$

$$= \frac{2}{3}(3\sqrt{3} - 2\sqrt{2}).$$

Uppgift 3.3 (Sid. 2)

Lösning

$$\underline{z = 1 - x - y;} \quad \underline{D: x^2 + y^2 \leq 1.}$$

$S: r(u,v) = (u, v, 1-u-v)$, $(u,v) \in D$; forts

$$r'_u \times r'_v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1,1,1) \Rightarrow d\sigma = |r'_u \times r'_v| \, du dv =$$

$$= \sqrt{3}\, du dv \Rightarrow \mathcal{A}(S) = \iint_D d\sigma = \sqrt{3} \cdot \mathcal{A}(D) = \underline{\sqrt{3}\cdot\pi.}$$

Uppgift 3.4 (Sid. 2)

Lösning

$$\underline{z = f(x,y) = 1 - x^2 - y^2;} \quad \underline{D: x^2 + y^2 \leq 1.}$$

Observera att $z \geq 0 \Leftrightarrow 1 - x^2 - y^2 \geq 0 \Leftrightarrow \underline{x^2 + y^2 \leq 1.}$

$$\begin{cases} x = u\cos v \\ y = u \sin v \end{cases} \Rightarrow z = f(u\cos v, u \sin v) = \underline{1 - u^2} \Rightarrow$$

$$\Rightarrow \underline{S: r(u,v) = (u\cos v, u \sin v, 1-u^2);} \quad \Delta: \begin{cases} 0 \leq u \leq 1 \\ 0 \leq v < 2\pi \end{cases}.$$

$$\begin{cases} r'_u = (\cos v, \sin v, -2u) \\ r'_v = (-u\sin v, u\cos v, 0) \end{cases} \Rightarrow r'_u(u,v) \times r'_v(u,v) =$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (2u^2\cos v, 2u^2 \sin v, u) \Rightarrow$$

$$d\sigma = |r'_u \times r'_v| \, du dv = \sqrt{4u^4 + u^2}\, du dv = \sqrt{4u^2+1}\, u\, du dv$$

$$\Rightarrow \mathcal{A}(S) = \iint_\Delta d\sigma = \int_0^1 \sqrt{4u^2+1}\, u\, du \int_0^{2\pi} dv = /t = 4u^2+1/$$

$= 2\pi \int_1^5 \frac{1}{8}\sqrt{t}\,dt = \frac{\pi}{4}\left[\frac{2}{3}t^{3/2}\right]_1^5 = \underline{\frac{\pi}{6}(5\sqrt{5}-1)}.$

Anm. Att finna en bra parametrisering är fråga om rutin.

Uppgift 3.5 (Sid. 2)

Lösning

$S: \mathbf{r}(u,v) = (u\cos v, u\sin v, u^2);\ \Delta = [0,1]^2.$

$\mathbf{r}'_u \times \mathbf{r}'_v = \hat{x}\frac{\partial(y,z)}{\partial(u,v)} + \hat{y}\frac{\partial(z,x)}{\partial(u,v)} + \hat{z}\frac{\partial(x,y)}{\partial(u,v)} =$

$= (-2u^2\cos v, -2u^2\sin v, u) \Rightarrow$

$d\sigma = |\mathbf{r}'_u \times \mathbf{r}'_v|\,du\,dv = \sqrt{4u^2+1}\,u\,du\,dv;\ (u,v)\in\Delta;$

$\mathcal{A}(S) = \iint_\Delta d\sigma = \int_0^1 \sqrt{4u^2+1}\,u\,du \int_0^1 dv \overset{!}{=} \underline{\frac{1}{12}(5\sqrt{5}-1)}.$

$\mathcal{I} \overset{!}{=}$ underförstås integrationen i 3.4.

Uppgift 3.6 (Sid. 2)

Lösning

$S: \mathbf{r}(u,v) = (u^2, v^2, \sqrt{2}uv);\ \Delta: u^2+v^2 \le 1.$

$\mathbf{r}'_u \times \mathbf{r}'_v = (2u, 0, \sqrt{2}v) \times (0, 2v, \sqrt{2}u) = (-2\sqrt{2}u^2, -2\sqrt{2}v^2, -4uv)$

$\Rightarrow d\sigma = |\mathbf{r}'_u \times \mathbf{r}'_v|\,du\,dv = \sqrt{8u^4+8v^4+16u^2v^2}\,du\,dv =$

$= \sqrt{8(u^2+v^2)^2}\,du\,dv = \sqrt{8}\cdot(u^2+v^2)\,du\,dv \Rightarrow$

$\mathcal{A}(S) = \sqrt{8}\iint_\Delta (u^2+v^2)\,du\,dv = \left[\begin{array}{l|l} u = r\cos\theta & 0\le r\le 1 \\ v = r\sin\theta & 0\le\theta\le 2\pi \end{array}\right] =$

$= \sqrt{8}\int_0^1 r^3\,dr \int_0^{2\pi} d\theta = 2\sqrt{2}\cdot\frac{1}{4}\cdot 2\pi = \underline{\sqrt{2}\pi}.$

Uppgift 3.7 (Sid. 2)

Lösning

$S: \mathbf{r}(u,v) = (2u\cosh v, 2u\sinh v, 2u);\ \Delta: \begin{cases} 0\le u\le 1 \\ 0\le v\le 1 \end{cases};$

$\mathbf{r}'_u \times \mathbf{r}'_v = 2(\cosh v, \sinh v, 1) \times (2u\sinh v, 2u\cosh v, 0)$

$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2\cosh v & 2\sinh v & 2 \\ 2u\sinh v & 2u\cosh v & 0 \end{vmatrix} =$

$= (-4u\cosh v, 4u\sinh v, 4u) \Rightarrow$

$d\sigma = |\mathbf{r}'_u \times \mathbf{r}'_v|\,du\,dv = 4u\sqrt{\cosh^2 v + \sinh^2 v + 1}\,du\,dv$

$= 4u\sqrt{\cosh 2v + 1}\,du\,dv = 4\sqrt{2}\sqrt{\cosh^2 v}\,u\,du\,dv$

$= 2\sqrt{2}\cdot 2u\,du\cdot\cosh v\,dv$ (separerat) $\Rightarrow$

$$\mathcal{A}(S) = \iint_\Delta d\sigma = 2\sqrt{2}\int_0^1 2u\,du \cdot \int_0^1 \cosh v\,dv =$$

$$= 2\sqrt{2}\cdot [u^2]_0^1 \cdot [\sinh v]_0^1 =$$

$$= 2\sqrt{2}\cdot 1 \cdot \sinh 1 = \underline{\sqrt{2}(e-e^{-1})}.$$

Några fakta om de hyperboliska funktionerna:

$\cosh x = \frac{e^x+e^{-x}}{2}$, $\sinh x = \frac{e^x-e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$.

$\cosh^2 x - \sinh^2 x = 1$ (den hyperboliska ettan).

$\cosh^2 x + \sinh^2 x = \cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$.

$\underline{D\sinh x} = \cosh x$, $\underline{D\cosh x} = \sinh x$.

Uppgift 3.8 (Sid. 2)

Lösning

$\underline{S: r(u,v) = (ue^v, ue^{-v}, \sqrt{2}u)}$; $\Delta: \begin{cases} 0 \le u \le \sqrt{2} \\ 0 \le v \le \frac{1}{2}\ln 2 \end{cases}$;

$$r'_u \times r'_v = (e^v, e^{-v}, \sqrt{2}) \times (ue^v, -ue^{-v}, 0) =$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ e^v & e^{-v} & \sqrt{2} \\ ue^v & -ue^{-v} & 0 \end{vmatrix} = (\sqrt{2}ue^{-v}, \sqrt{2}ue^v, -2u) \Rightarrow$$

$$d\sigma = |r'_u \times r'_v|\,du\,dv = \sqrt{2}u \cdot \sqrt{e^{-2v}+e^{2v}+2}\,du\,dv =$$

$$= \sqrt{2}u\sqrt{2\cosh 2u + 2}\,du\,dv = 2u\sqrt{\cosh 2u + 1}\,du\,dv =$$

$$= 2\sqrt{2}u \cdot \sqrt{\cosh^2 v}\,du\,dv = \sqrt{2}\cdot 2u\,du\cdot \cosh v\,dv \Rightarrow$$

$$\mathcal{A}(S) = \sqrt{2}\int_0^{\sqrt{2}} 2u\,du \cdot \int_0^{\frac{1}{2}\ln 2} \cosh v\,dv = \sqrt{8}\sinh\left(\tfrac{1}{2}\ln 2\right) =$$

$$= \sqrt{8}\cdot\frac{1}{2}\left(\underline{e^{\frac{1}{2}\ln 2} - e^{-\frac{1}{2}\ln 2}}\right) = \sqrt{2}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right) = \underline{1}.$$

Ann. $e^{\frac{1}{2}\ln 2} = e^{\ln 2^{1/2}} = 2^{1/2} = \underline{\sqrt{2}}$.

Uppgift 3.9 (Sid. 3)

Lösning

(1) $x^2+y^2+z^2 = 1 \wedge z \ge 0 \Leftrightarrow \underline{z = \sqrt{1-x^2-y^2}}$, $\underline{x^2+y^2 \le 1}$.

$r(x,y) = (x, y, \sqrt{1-x^2-y^2})$; $\Delta: x^2+y^2 \le 1$.

$$r'_x \times r'_y = \left(1, 0, -\frac{x}{z}\right) \times \left(0, 1, -\frac{y}{z}\right) = \left(\frac{x}{z}, \frac{y}{z}, 1\right) \Rightarrow$$

$$d\sigma = |r'_x \times r'_y|\,dx\,dy = \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\,dx\,dy = \frac{dx\,dy}{z};$$

$$\mathcal{A}(S) = \iint_\Delta d\sigma = \iint_\Delta \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy = /\text{polärt}/ =$$

$$= \left[\begin{array}{l|l} x = r\cos\theta & 0 \le r \le 1 \\ y = r\sin\theta & 0 \le \theta \le 2\pi \end{array}\right] = \int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr \int_0^{2\pi} d\theta =$$

$$= \left[-\sqrt{1-r^2}\right]_0^1 \cdot [\theta]_0^{2\pi} = \underline{2\pi}.$$

(2) $S: \mathbf{r}(u,v)=(\sin u\cos v, \sin u\sin v, \cos v)$; $\Delta: \begin{cases} 0\le u\le 2\pi \\ 0\le v\le \frac{\pi}{2} \end{cases}$;

$$\begin{cases} \mathbf{r}'_u=(\cos u\cos v, \cos u\sin v, 0) \\ \mathbf{r}'_v=(-\sin u\sin v, \sin u\cos v, -\sin v) \end{cases} \Rightarrow \mathbf{r}'_u\times\mathbf{r}'_v=$$

$$=\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos u\cos v & \cos u\sin v & 0 \\ -\sin u\sin v & \sin u\cos v & -\sin v \end{vmatrix}=$$

$$=(-\cos u\sin^2 v,\ \cos u\sin v\cos v,\ \sin u\cos u)=$$

$$=\cos u\cdot(-\sin^2 u,\ \tfrac{1}{2}\sin 2v,\ \sin u)\Rightarrow$$

$d\sigma=|\mathbf{r}'_u\times\mathbf{r}'_v|\,du\,dv=\sin v\,du\,dv$, $(u,v)\in\Delta$;

$$\mathcal{A}(S)=\iint_\Delta d\sigma=\int_0^{2\pi}du\cdot\int_0^{\pi/2}\sin v\,dv=\underline{2\pi}.$$

Anm. Areaelementet på sfären $r=R$ är

$$d\sigma=R^2\sin\theta\,d\theta\,d\varphi.$$

Uppgift 3.10 (Sid. 3)

Lösning

$$\begin{cases} S: \mathbf{r}(u,v)=(u\cos v, u\sin v, h(1-\frac{u}{R})) \\ \Delta: 0\le u\le R,\ 0\le v\le 2\pi. \end{cases};$$

$$\mathbf{r}'_u\times\mathbf{r}'_v=(\cos v,\sin v,-\tfrac{h}{R})\times(-u\sin v, u\cos v, 0)=$$

$$=\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos v & \sin v & -h/R \\ -u\sin v & u\cos v & 0 \end{vmatrix}=$$

$$=(\tfrac{h}{R}u\cos v,\ \tfrac{h}{R}u\sin v,\ u)\Rightarrow$$

$d\sigma=|\mathbf{r}'_u\times\mathbf{r}'_v|\,du\,dv=u\sqrt{(h/R)^2+1}\,du\,dv$, $(u,v)\in\Delta$;

$$\mathcal{A}(S)=\iint_\Delta d\sigma=\frac{1}{R}\sqrt{h^2+R^2}\int_0^R u\,du\cdot\int_0^{2\pi}dv=$$

$$=\frac{1}{R}\cdot\sqrt{h^2+R^2}\cdot\frac{1}{2}R^2\cdot 2\pi=\underline{\pi R\sqrt{h^2+R^2}}.$$

Uppgift 3.11 (Sid. 3)

Lösning

$S: \mathbf{r}(u,v)=((R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u)$

$$\left.\begin{aligned} \mathbf{r}'_u&=(-r\sin u\cos v, -r\sin u\sin v, r\cos u) \\ \mathbf{r}'_v&=(-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0) \end{aligned}\right\}\Rightarrow$$

$$\mathbf{r}'_u\times\mathbf{r}'_v=-r(R+r\cos u)\cos u\cos v\,\hat{x}-$$

$$-r(R+r\cos u)\cos u\sin v\,\hat{y}-$$

$$-r(R+r\cos u)\sin u\,\hat{z}=$$

$$=\underline{-r(R+r\cos u)\cdot(\cos u\cos v, \cos u\sin v, \sin u)}$$

$d\sigma = |r'_u \times r'_v| du dv = r\cdot(R+r\cos u) \Rightarrow \mathcal{A}(S) =$

$= \int_0^{2\pi} dv \cdot \int_0^{2\pi} r(R+r\cos u)\,du = \int_0^{2\pi} dv \int_0^{2\pi} rR\,du = \underline{4\pi^2 rR}.$

Anm. Ovanstående yta är en torus.

Uppgift 3.12 (Sid. 3)

Lösning

$S: r(u,v) = (u\cos v,\ u\sin v,\ \frac{1}{2}u^2);\ \Delta: \begin{cases} 0\le u\le 1 \\ 0\le v\le \frac{\pi}{2} \end{cases};$

$\begin{cases} r'_u = (\cos v,\ \sin v,\ u) \\ r'_v = (-u\sin v,\ u\cos v,\ 0) \end{cases} \Rightarrow r'_u \times r'_v = (-u\cos v, -u\sin v, 1)u$

$\Rightarrow d\sigma = |r'_u \times r'_v| du dv = \sqrt{u^2+1}\,u\,du\,dv,\ (u,v)\in\Delta;$

$\mathcal{A}(S) = \int_0^1 \sqrt{u^2+1}\,u\,du \int_0^{\pi/2} dv = /t = u^2+1 \Rightarrow dt = 2u\,du/ =$

$= \frac{\pi}{4}\int_1^2 \sqrt{t}\,dt := \frac{\pi}{4}\left[\frac{2t\sqrt{t}}{3}\right]_1^2 = \underline{\frac{\pi}{6}(2\sqrt{2}-1)}.$

Annan lösning

$S: r(x,y) = (x, y, \frac{1}{2}(x^2+y^2));\ \Delta: x^2+y^2\le 1;\ x,y\ge 0.$

$r'_x \times r'_y = (1,0,x)\times(0,1,y) = (-x,-y,1) \Rightarrow |r'_x \times r'_y| =$

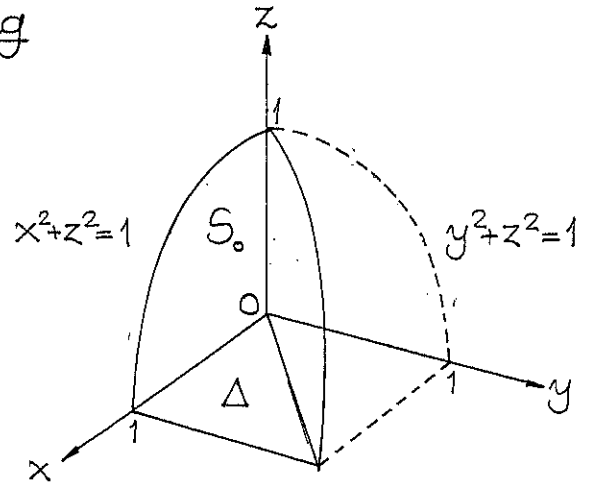
$= \sqrt{x^2+y^2+1} \Rightarrow d\sigma = \sqrt{x^2+y^2+1}\,dx\,dy \Rightarrow \mathcal{A}(S) =$

$= \iint_\Delta \sqrt{x^2+y^2+1}\,dx\,dy = \left[\begin{array}{l|l} x = r\cos\theta & 0\le r\le 1 \\ y = r\sin\theta & 0\le\theta\le\frac{\pi}{2} \end{array}\right] =$

$= \int_0^1 \sqrt{r^2+1}\cdot r\,dr \int_0^{\pi/2} d\theta = \frac{\pi}{2}\cdot\left[\frac{1}{3}(r^2+1)^{3/2}\right]_0^1 = \frac{\pi}{6}(2\sqrt{2}-1).$

Uppgift 3.13 (Sid. 3)

Lösning



I figuren ovan syns den del av ytan som ligger i den första oktanten; dess projektion i xy-planet är $\Delta: 0\le x\le 1,\ 0\le y\le x.$

$S_0: r(x,y) = (x, y, \sqrt{1-x^2});\ \Delta: \begin{cases} 0\le x\le 1 \\ 0\le y\le x \end{cases};$ forts.

$$\left.\begin{matrix} r_x' = (1, 0, -\frac{x}{\sqrt{1-x^2}}) \\ r_y' = (0, 1, 0) \end{matrix}\right\} \Rightarrow r_x' \times r_y' = (\frac{x}{\sqrt{1-x^2}}, 0, 1) \Rightarrow$$

$$\Rightarrow d\sigma = |r_x' \times r_y'| dx dy = \frac{1}{\sqrt{1-x^2}} dx dy \Rightarrow \mathcal{A}(S) =$$

$$= \iint_\Delta \frac{dx dy}{\sqrt{1-x^2}} = \int_0^1 dx \frac{1}{\sqrt{1-x^2}} \int_0^x dy = \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = 1.$$

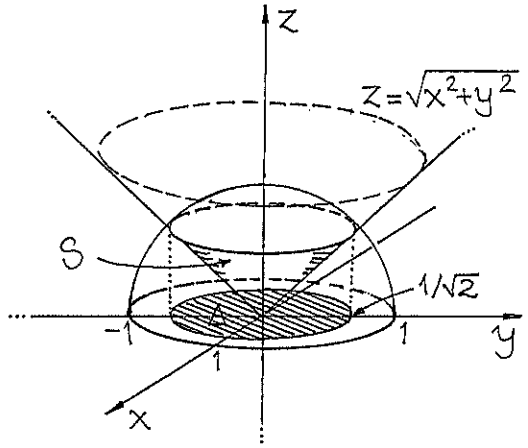
Svar: Den sökta arean är 4.

Uppgift 3.14 (Sid. 3)

Lösning

$$z = \sqrt{x^2+y^2} \Rightarrow x^2+y^2+z^2 = 2(x^2+y^2) = 1 \Leftrightarrow$$

$$\Leftrightarrow \Delta : x^2+y^2 \leq \frac{1}{2}.$$



$S : r(x,y) = (x, y, \sqrt{x^2+y^2})$; $\Delta : \sqrt{x^2+y^2} \leq 1/\sqrt{2}$.

$$\left\{\begin{matrix} r_x' = (1, 0, \frac{x}{z}) \\ r_y' = (0, 1, \frac{y}{z}) \end{matrix}\right. \Rightarrow r_x' \times r_y' = (-\frac{x}{z}, -\frac{y}{z}, 1) \Rightarrow |r_x' \times r_y'| =$$

$$= \sqrt{2} \Rightarrow \mathcal{A}(S) = \sqrt{2} \iint_\Delta dx dy = \sqrt{2} \cdot \frac{\pi}{2} = \underline{\frac{\pi}{\sqrt{2}}}.$$

4. Nablakalkyl

Uppgift 4.1 (Sid. 3)

Lösning

$$\underline{A(x,y,z) = (2x-6y+z^2,\ 4x-y^2+2z,\ x^2+y^2-2z^2).}$$

a) $\nabla \cdot A = \partial_x(2x-6y+z^2) + \partial_y(4x-y^2+2z) + \partial_z(x^2+y^2-2z^2)$

$$= \underline{2-2y-4z.}$$

b) $$\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 2x-6y+z^2 & 4x-y^2+2z & x^2+y^2-2z^2 \end{vmatrix} =$$

$$= (\partial_y(x^2+y^2-2z^2) - \partial_z(4x-y^2+2z))\,\hat{x} +$$

$$+ (\partial_z(2x-6y+z^2) - \partial_x(x^2+y^2-2z^2))\,\hat{y} +$$

$$+ (\partial_x(4x-y^2+2z) - \partial_y(2x-6y+z^2))\,\hat{z} =$$

$$= (2y-2,\ 2z-2x,\ 4-6) = \underline{(2y-2,\ 2z-2x,\ 2).}$$

c) $\nabla(\nabla\cdot\mathbb{A}) = \nabla(2-2y-4z) = \underline{(0,-2,-4)}$.

d) $\nabla\times(\nabla\times\mathbb{A}) = \nabla\times(2y-2, 2z-2x, 2) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 2y-2 & 2z-2x & 2 \end{vmatrix}$

$= (\partial_y(2)-\partial_z(2z-2x))\hat{x} + (\partial_z(2y-2)-\partial_x(2))\hat{y} + (\partial_x(2z-2x)-$

$-\partial_y(2y-2))\hat{z} = \underline{(-2,0,-4)}$.

Anm. $\partial_x = \frac{\partial}{\partial x} = D_x$ osv.

Uppgift 4.2 (Sid. 3)

Lösning

a) $\nabla\Phi = (\partial_x\Phi, \partial_y\Phi) = (2x+y, 2y+x) \Leftrightarrow \begin{cases} \partial_x\Phi = 2x+y & (1) \\ \partial_y\Phi = x+2y & (2) \end{cases}$

$(1) \Rightarrow \Phi = x^2+xy+f(y) \Rightarrow \partial_y\Phi = x+f'(y) \overset{(2)}{=} x+2y \Leftrightarrow$

$f'(y) = 2y \Leftrightarrow f(y) = y^2+C \overset{!}{\Rightarrow} \underline{\Phi(x,y) = x^2+xy+y^2+C}$.

b) $\nabla\Phi = (2x\sin x^2, \cos y) \Leftrightarrow \begin{cases} \partial_x\Phi = 2x\sin x^2 & (1) \\ \partial_y\Phi = \cos y & (2) \end{cases}$;

$(1) \Rightarrow \Phi = -\cos x^2 + f(y) \Rightarrow \partial_y\Phi = f'(y) \overset{(2)}{=} \cos y \Leftrightarrow f(y) =$

$= \sin y + C \Rightarrow \underline{\Phi(x,y) = -\cos x^2 + \sin y + C}$.

c) $\nabla\Phi = (e^{xy}, e^{xy}) \Leftrightarrow \begin{cases} \partial_x\Phi = e^{xy} \\ \partial_y\Phi = e^{xy} \end{cases} \Leftrightarrow \begin{cases} \Phi = \frac{e^{xy}}{y} + f(y) \\ \Phi = \frac{e^{xy}}{x} + g(x) \end{cases}$;

Det existerar inte någon Φ; $\partial_x e^{xy} \neq \partial_y e^{xy}$.

d) $\nabla\Phi = ((1+x+xz)e^{x+y+xz}, xe^{x+y+xz}, x^2e^{x+y+xz})$

$$\Leftrightarrow \begin{cases} \partial_x\Phi = (1+x+xz)e^{x+y+xz} & (1) \\ \partial_y\Phi = xe^{x+y+xz} & (2) \\ \partial_z\Phi = x^2e^{x+y+xz} & (3) \end{cases}$$

$(2) \Rightarrow \Phi = xe^{x+y+xz} + f(x,z) \Rightarrow \partial_z\Phi = x^2e^{x+y+xz} +$

$+\frac{\partial f}{\partial z} \overset{(3)}{=} x^2e^{x+y+xz} \Leftrightarrow \partial_z f(x,z) = 0 \Leftrightarrow f(x,z) = g(x) \Rightarrow$

$\Phi = xe^{x+y+xz} + g(x) \Rightarrow \partial_x\Phi = (1+x+xz)e^{x+y+xz} +$

$+ g'(x) \overset{(1)}{=} (1+x+xz)e^{x+y+xz} \Leftrightarrow g'(x) = 0 \Leftrightarrow g(x) = C$

$\Rightarrow \underline{\Phi(x,y,z) = xe^{x+y+xz} + C}$.

e) $\nabla\Phi = (\sin yz, xz\cos yz, xy\sin yz) \Leftrightarrow$

$\underline{\partial_x\Phi = \sin yz}$ (1), $\underline{\partial_y\Phi = xz\cos yz}$ (2); $\underline{\partial_z = xy\sin yz}$ (3).

$(3) \Rightarrow \Phi = -x\cos yz + f(x,y) \Rightarrow \partial_y\Phi = xz\sin yz + \partial_y f =$

$\overset{(2)}{=} xz\cos yz \Leftrightarrow \partial_y f(x,y) = 0 \Leftrightarrow f(x,y) = g(x) \Rightarrow$

$\Phi = -x\cos yz + g(x) \Rightarrow \partial_x \Phi = -\cos yz + g'(x) = (1) =$

$= \sin yz \Leftrightarrow g'(x) \overset{!}{=} \sin yz + \cos yz$, motsägelse;

Det existerar ingen Φ i detta fall.

f) $\nabla\Phi = (2xy+z^2,\ x^2+2yz,\ y^2+2xz) \Leftrightarrow$

$\Leftrightarrow \partial_x\Phi = 2xy+z^2$ (1), $\partial_x\Phi = x^2+2yz$ (2), $\partial_z\Phi = y^2+2xz$ (3)

(1) $\Rightarrow \Phi = x^2y + xz^2 + f(y,z) \Rightarrow \partial_y\Phi = x^2 + \partial_y f \overset{(2)}{=} x^2+2yz$

$\Leftrightarrow \partial_y f(y,z) = 2yz \Leftrightarrow f(y,z) = y^2z + g(x) \Rightarrow \Phi =$

$= x^2y + xz^2 + y^2z + g(x) \Rightarrow \partial_x\Phi = 2xy + z^2 + g'(x) \overset{(1)}{=}$

$= 2xy + z^2 \Leftrightarrow g'(x) = 0 \Leftrightarrow g(x) = C$ (konstant) $\Rightarrow$

$\Phi(x,y,z) = x^2y + xz^2 + y^2z + C.$

Uppgift 4.3 (Sid. 3)

Lösning

$$\mathbb{A} = \left(\frac{xz}{x^2+y^2}, \frac{yz}{x^2+y^2}, 0\right).$$

$$\left.\begin{aligned} A_x = \frac{xz}{x^2+y^2} &\Rightarrow \partial_x A_x = \frac{z(x^2+y^2)-2x^2z}{(x^2+y^2)^2} = \frac{zy^2-zx^2}{(x^2+y^2)^2} \\ A_y = \frac{yz}{x^2+y^2} &\Rightarrow \partial_y A_y = \frac{z(x^2+y^2)-2y^2z}{(x^2+y^2)^2} = \frac{zx^2-zy^2}{(x^2+y^2)^2} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \operatorname{div}\mathbb{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z = \frac{zy^2-zx^2+zx^2-zy^2}{(x^2+y^2)^2} = 0.$$

Uppgift 4.4 (Sid. 3)

Lösning

$$\mathbb{A} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

$$\operatorname{rot}\mathbb{A} = (\partial_y A_z - \partial_z A_y,\ \partial_z A_x - \partial_x A_z,\ \partial_x A_y - \partial_y A_x) =$$

$$= (0-0,\ 0-0,\ \partial_x A_y - \partial_y A_x) = (0,0,\partial_x A_y - \partial_y A_x)$$

$$= \left(0, 0, \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) =$$

$$= \left(0, 0, \frac{2(x^2+y^2)-2(x^2+y^2)}{(x^2+y^2)^2}\right) = (0,0,0) = \underline{0}.$$

Uppgift 4.5 (Sid. 3)

Lösning

$$\mathbb{F}(x,y,z) = (yz, zx, xy)$$

$$\operatorname{rot}\mathbb{F} = (\partial_y F_z - \partial_z F_y,\ \partial_z F_x - \partial_x F_z,\ \partial_x F_y - \partial_y F_x) =$$

$$= (x-x,\ y-y,\ z-z) = (0,0,0) = \underline{0}.$$

Uppgift 4.6 (Sid. 3)

Lösning

Se nästföljande sida.

a) $\nabla\cdot(\Phi\mathbb{A}) = \nabla\cdot(\Phi(A_x, A_y, A_z)) = \nabla\cdot(\Phi A_x, \Phi A_y, \Phi A_z) =$

$= \partial_x(\Phi A_x) + \partial_y(\Phi A_y) + \partial_z(\Phi A_z) =$

$= (\partial_x\Phi)A_x + \Phi\partial_x A_x + (\partial_y\Phi)A_y + \Phi\partial_y A_y +$

$+ (\partial_z\Phi)A_z + \Phi\partial_z A_z =$

$= (\partial_x\Phi)A_x + (\partial_y\Phi)A_y + (\partial_z\Phi)A_z +$

$+ \Phi(\partial_x A_x + \partial_y A_y + \partial_z A_z) =$

$= (\partial_x\Phi, \partial_y\Phi, \partial_z\Phi)\cdot(A_x, A_y, A_z) +$

$+ \Phi(\partial_x, \partial_y, \partial_z)\cdot(A_x, A_y, A_z) =$

$= (\partial_x, \partial_y, \partial_z)\Phi\cdot(A_x, A_y, A_z) +$

$+ \Phi(\partial_x, \partial_y, \partial_z)\cdot(A_x, A_y, A_z) =$

$= \underline{(\nabla\Phi)\mathbb{A} + \Phi(\nabla\cdot\mathbb{A})}.$

Enhetsvektorerna $\hat{x}, \hat{y}, \hat{z}$ bör undvikas.

b) $\nabla\times(\Phi\mathbb{A}) = \nabla\times(\Phi A_x, \Phi A_y, \Phi A_z) =$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \Phi A_x & \Phi A_y & \Phi A_z \end{vmatrix} =$$

$= (\partial_y(\Phi A_z) - \partial_z(\Phi A_y), \partial_z(\Phi A_x) - \partial_x(\Phi A_z), \partial_x(\Phi A_y) - \partial_y(\Phi A_x))$

$= ((\partial_y\Phi)A_z + \Phi\partial_y A_z - (\partial_z\Phi)A_y - \Phi\partial_z A_y)\hat{x} +$

$+ ((\partial_z\Phi)A_x + \Phi\partial_z A_x - (\partial_x\Phi)A_z - \Phi\partial_x A_z)\hat{y} +$

$+ ((\partial_x\Phi)A_y + \Phi\partial_x A_y - (\partial_y\Phi)A_x - \Phi\partial_y A_x)\hat{z} =$

$= ((\partial_y\Phi)A_z - (\partial_z\Phi)A_y, (\partial_z\Phi)A_x - (\partial_x\Phi)A_z, (\partial_x\Phi)A_y - (\partial_y\Phi)A_x)$

$+ \Phi(\partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x) =$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x\Phi & \partial_y\Phi & \partial_z\Phi \\ A_x & A_y & A_z \end{vmatrix} + \Phi\nabla\times\mathbb{A} =$$

$= \underline{(\nabla\Phi)\times\mathbb{A} + \Phi\nabla\times\mathbb{A}}.$

c) $\nabla\cdot(\mathbb{A}\times\mathbb{B}) = \nabla\cdot(A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x) =$

$= \partial_x(A_yB_z - A_zB_y) + \partial_y(A_zB_x - A_xB_z) + \partial_z(A_xB_y - A_yB_x)$

$= \partial_x(A_yB_z) - \partial_x(A_zB_y) + \partial_y(A_zB_x) - \partial_y(A_xB_z) +$

$+ \partial_z(A_xB_y) - \partial_z(A_yB_x) =$

$= (\partial_xA_y)B_z + A_y(\partial_xB_z) - (\partial_xA_z)B_y - A_z(\partial_xB_y) +$

$+ (\partial_yA_z)B_x + A_z(\partial_yB_x) - (\partial_yA_x)B_z - A_x(\partial_yB_x +$

$+ (\partial_zA_x)B_y + A_x(\partial_zB_y) - (\partial_zA_y)B_x - A_y(\partial_zB_y) =$

$= (\partial_y A_z - \partial_z A_y) B_x + (\partial_z A_x - \partial_x A_z) B_y + (\partial_x A_y - \partial_y A_x) B_z$

$- (\partial_y B_z - \partial_z B_y) A_x + (\partial_z B_x - \partial_x B_z) A_y + (\partial_x B_y - \partial_y B_x) A_z$

$= (\nabla \times \mathbb{A}) \cdot \mathbb{B} - (\nabla \times \mathbb{B}) \cdot \mathbb{A} = \underline{(\nabla \times \mathbb{A}) \cdot \mathbb{B} - \mathbb{A} \cdot (\nabla \times \mathbb{B})}.$

d) $\nabla \times (\nabla \phi) = \nabla \times (\partial_x \Phi, \partial_y \Phi, \partial_z \Phi) = (\partial_y \partial_z \Phi - \partial_z \partial_y \Phi)\hat{x} +$

$+ (\partial_z \partial_x \Phi - \partial_x \partial_z \phi)\hat{y} + (\partial_x \partial_y \Phi - \partial_y \partial_x \Phi)\hat{z} = (0,0,0) = \mathbb{0}.$

Anm. $f \in \mathcal{C}^2 \Rightarrow \partial_i \partial_j f = \partial_j \partial_i f$, $i \neq j$.

e) $\nabla \cdot (\nabla \times \mathbb{A}) = \nabla \cdot (\partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x) =$

$= \partial_x (\partial_y A_z - \partial_z A_y) + \partial_y (\partial_z A_x - \partial_x A_z) + \partial_z (\partial_x A_y - \partial_y A_x) =$

$= \partial_x \partial_y A_z - \partial_x \partial_z A_y + \partial_y \partial_z A_x - \partial_y \partial_x A_z + \partial_z \partial_x A_y - \partial_z \partial_y A_x =$

$= \partial_x \partial_y A_z - \partial_y \partial_x A_z + \partial_y \partial_z A_x - \partial_z \partial_y A_x + \partial_x \partial_z A_y - \partial_z \partial_x A_y =$

$= (\partial_x \partial_y - \partial_y \partial_x) A_z + (\partial_y \partial_z - \partial_z \partial_y) A_x + (\partial_x \partial_z - \partial_z \partial_x) A_y = 0.$

Jfr anmärkningen ovan.

Uppgift 4.7 (Sid. 4)

Lösning

a) $\nabla (\mathbb{a} \cdot \mathbb{r}) = \nabla (a_x \cdot x + a_y \cdot y + a_z \cdot z) = \hat{x} \partial_x (a_x \cdot x + a_y \cdot y + a_z \cdot z) +$

$+ \hat{y} \partial_y (a_x \cdot x + a_y \cdot y + a_z \cdot z) + \hat{z} \partial_z (a_x \cdot x + a_y \cdot y + a_z \cdot z) =$

$= (a_x, a_y, a_z) = \mathbb{a}.$

b) $\nabla \cdot \mathbb{r} = (\partial_x, \partial_y, \partial_z) \cdot (x, y, z) = \partial_x (x) + \partial_y (y) + \partial_z (z) = 1 + 1 + 1 = 3.$

c) $r^2 = x^2 + y^2 + z^2 \Rightarrow \partial_x r^2 = 2x \Rightarrow 2r \partial_x r = 2x \Rightarrow \partial_x r = \frac{x}{r}$

och p.s.s. $\partial_y r = \frac{y}{r}$ och $\partial_z r = \frac{z}{r}$ s.a. $\nabla r = \frac{\mathbb{r}}{r} = \hat{r} \Rightarrow$

$\nabla \Phi(r) = (\partial_x, \partial_y, \partial_z) \Phi(r) = (\partial_x \Phi(r), \partial_y \Phi(r), \partial_z \Phi(r)) =$

$= (\Phi'(r) \partial_x r, \Phi'(r) \partial_y r, \Phi'(r) \partial_z r) =$

$= \Phi'(r) (\partial_x r, \partial_y r, \partial_z r) = \Phi'(r) \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) =$

$= \Phi'(r) \cdot \frac{1}{r} (x, y, z) = \frac{d\Phi}{dr} \frac{\mathbb{r}}{r}.$

Anm. $\nabla f(g(x,y,z)) = f'(g(x,y,z)) \cdot \nabla g(x,y,z).$

d) $\nabla \cdot (\Phi(r) \mathbb{r}) = (\nabla \Phi(r)) \cdot \mathbb{r} + \Phi(r) \nabla \cdot \mathbb{r} = \left(\frac{d\Phi}{dr}\right) \frac{\mathbb{r}}{r} \cdot \mathbb{r} + \Phi(r) \cdot 3 =$

$= \frac{d\Phi}{dr} \frac{\mathbb{r} \cdot \mathbb{r}}{r} + 3\Phi(r) = \frac{d\Phi}{dr} \cdot \frac{r^2}{r} + 3\Phi(r) = r \frac{d\Phi}{dr} + 3\Phi(r).$

Anm. Se 4.6 a), b) och c).

e) $\nabla \cdot (\mathbb{a} \times \mathbb{r}) = (\nabla \times \mathbb{a}) \cdot \mathbb{r} - \mathbb{a} \cdot (\nabla \times \mathbb{r}) = \mathbb{0} \cdot \mathbb{r} - \mathbb{a} \cdot \mathbb{0} = 0.$

f) $\nabla \times (\Phi(r) \mathbb{r}) = \underbrace{(\nabla \Phi(r))}_{\Phi'(r) \cdot \hat{r}} \times \mathbb{r} + \Phi(r) \underbrace{(\nabla \times \mathbb{r})}_{=\mathbb{0}} = \mathbb{0} + \mathbb{0} = \mathbb{0}.$

g) $\underline{(A\times B)\times C = B(A\cdot C) - A(B\cdot C)}$

$(r\times a)\times r = a(r\cdot r) - r(a\cdot r) = r^2 a - (a\cdot r)r$;

$\nabla\cdot((r\times a)\times r) = \nabla\cdot(r^2 a) - \nabla\cdot((a\cdot r)r) =$

$= (\nabla r^2)\cdot a + r^2\nabla\cdot a - (\nabla(a\cdot r))\cdot r - (a\cdot r)\nabla\cdot r$

$= (2r\cdot\frac{r}{r})\cdot a + r^2\cdot 0 - a\cdot r - 3(a\cdot r) =$

$= 2a\cdot r - a\cdot r - 3a\cdot r = \underline{-2a\cdot r}.$

h) $\nabla\times(a\times r) = \nabla\times(a_y z - a_z y,\ a_z x - a_x z,\ a_x y - a_y x) =$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ a_y z - a_z y & a_z x - a_x z & a_x y - a_y x \end{vmatrix} =$$

$= (\partial_y(a_x y - a_y x) - \partial_z(a_z x - a_x z))\hat{x} +$

$+ (\partial_z(a_y z - a_z y) - \partial_x(a_x y - a_y x))\hat{y} +$

$+ (\partial_x(a_z x - a_x z) - \partial_y(a_y z - a_z y))\hat{z} =$

$= (a_x + a_x,\ a_y + a_y,\ a_z + a_z) = \underline{2a}.$

Anm. Ett antal vektoridentiteter ska en ingenjör kunna utantill. De finns samlade i handboken BETA.

Uppgift 4.8 (Sid. 4)

Lösning: $\nabla^2 = (\partial_x)^2 + (\partial_y)^2 + (\partial_z)^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

$$VL = \nabla\times(\nabla\times F) = \nabla\times\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \nabla\times(-y,-z,-x) =$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ -y & -z & -x \end{vmatrix} = (1,1,1).$$

$HL = \nabla(\nabla\cdot F) - \nabla^2 F = \nabla(x+y+z) - 0 = (1,1,1)$.

Anm. $A\times(B\times C) = B(A\cdot C) - (A\cdot B)C$;

$\underline{A = B = \nabla} \Rightarrow \nabla\times(\nabla\times F) = \nabla(\nabla\cdot F) - \nabla^2 F$.

5. Kurvintegraler i $\mathbb{R}^2$. Greens formel

Uppgift 5.1 (Sid. 4)

Lösning: $\underline{A = (-x\cos y,\ y\sin x)}$; $\underline{dr = (dx, dy)}$

γ: $r(t) = (t, 2t)$, $0 \le t \le \pi$;

$$\int_\gamma A\cdot dr = \int_0^\pi (-t\cos 2t,\ 2t\sin t)\cdot(1,2)\,dt =$$

$$= \int_0^\pi \underbrace{(4t\sin t - t\cos 2t)}_{t(4\sin t - \cos 2t)}\,dt =$$

$= [t(-4\cos t - \frac{1}{2}\sin 2t)]_0^{\pi} + \underbrace{\int_0^{\pi}(4\cos t + \frac{1}{2}\sin 2t)dt}_{=0} = 4\pi.$

Uppgift 5.2 (Sid. 4)

Lösning

a) $y = x \Rightarrow \underline{\gamma: r(t) = (t,t),\ 0 \le t \le 1.}$

$\int_\gamma xy\,dx + (y-x)\,dy = \int_0^1 t^2\,dt = \frac{1}{3}.$

b) $y = \sqrt{x} \Rightarrow \underline{\gamma: r(t) = (t^2,t),\ 0 \le t \le 1.}$

$\int_\gamma xy\,dx + (y-x)\,dy = \int_0^1 (t^3 \cdot 2t + (t - t^2))\,dt =$

$= \int_0^1 (2t^4 - t^2 + t)\,dt = \left[\frac{2}{5}t^5 - \frac{1}{3}t^3 + \frac{1}{2}t^2\right]_0^1 = \frac{2}{5} - \frac{1}{3} + \frac{1}{2} = \frac{17}{30}.$

Anm. $\gamma: r = r(t),\ \alpha \le t \le \beta;$

$\int_\gamma A \cdot dr = \int_\alpha^\beta A(r(t)) \cdot \frac{dr}{dt}\,dt$

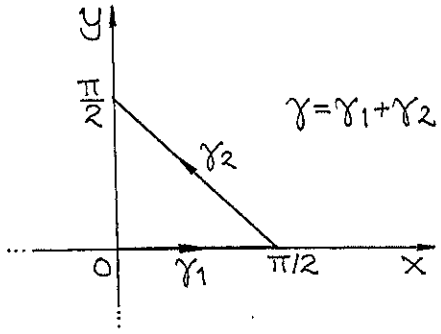
Se sidan 326 i Persson & Böiers.

Uppgift 5.3 (Sid. 4)

Lösning

$\omega = \cos y\,dx + \sin x\,dy$

Integrationsvägen är sammansatt: $\gamma = \gamma_1 + \gamma_2.$



$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$ (Se sidan 330 i P&B).

(1) $\underline{\gamma_1: r(t) = (t,0),\ 0 \le t \le \frac{\pi}{2}.}$

$\int_{\gamma_1} \omega = \int_0^{\pi/2} \omega(\gamma_1) = \int_0^{\pi/2} dt = \frac{\pi}{2};$

(2) $\gamma_2: r(t) = (\frac{\pi}{2}, 0) + (-\frac{\pi}{2}, \frac{\pi}{2})t = (\frac{\pi}{2} - \frac{\pi}{2}t, \frac{\pi}{2}t),\ 0 \le t \le 1$

$= \underline{(\frac{\pi}{2} - t, t),\ 0 \le t \le \frac{\pi}{2}}$

$\int_{\gamma_2} \omega = \int_0^{\pi/2} \omega(\gamma_2) = \int_0^{\pi/2} (\cos t \cdot (-1) + \sin(\frac{\pi}{2} - t))\,dt =$

$= \int_0^{\pi/2} (-\cos t + \cos t)\,dt = 0.$

Sammanfattningsvis fås $\int_\gamma \omega = \frac{\pi}{2}.$

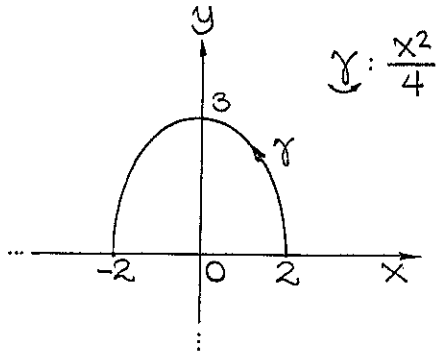
De beteckningar jag använder kommer från differentialgeometrin.

$\gamma: r = r(t),\ \alpha \le t \le \beta;\quad \omega = P(x,y)dx + Q(x,y)dy$

$\omega(\gamma) = P(x(t), y(t) \cdot x'(t)\,dt + Q(x(t), y(t))\,y'(t)\,dt$

Uppgift 5.4 (Sid. 4)

Lösning



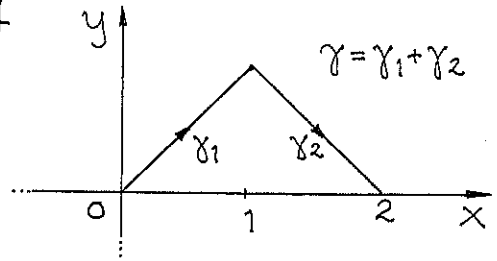
$\gamma: \frac{x^2}{4}+\frac{y^2}{9}=1,\ y\geq 0.$

$\gamma: r(t)=(2\cos t, 3\sin t),\ 0\leq t\leq \pi;\quad A=(y,-x).$

$$\int_\gamma A\cdot dr=\int_0^\pi A(r(t))\cdot r'(t)\,dt= \text{/Obs! } \omega=A\cdot dr/=$$
$$=\int_0^\pi (3\sin t,-2\cos t)\cdot(-2\sin t,3\cos t)\,dt=$$
$$=\int_0^\pi (-6\sin^2 t-6\cos^2 t)\,dt=\int_0^\pi (-6)\,dt=\underline{-6\pi}.$$

Uppgift 5.5 (Sid. 4)

Lösning



$\gamma_1: r(t)=(t,t),\ 0\leq t\leq 1;\quad \gamma_2: r(t)=(t,2-t),\ 1\leq t\leq 2.$

$$\int_\gamma F\cdot dr=\int_{\gamma_1} F\cdot dr+\int_{\gamma_2} F\cdot dr;$$

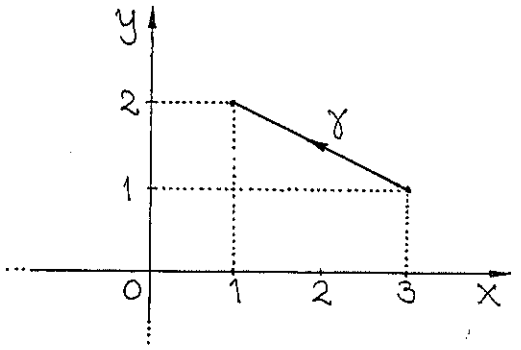
(1) $\int_{\gamma_1} F\cdot dr=\int_0^1 (2t^2,0)\cdot(1,1)\,dt=\int_0^1 2t^2\,dt=\frac{2}{3}.$

(2) $\int_{\gamma_2} F\cdot dr=\int_1^2 (t^2+(2-t)^2,\ t^2-(2-t)^2)\cdot(1,-1)\,dt=$
$$=\int_1^2 (2t^2-4t+4,\ 4t-4)\cdot(1,-1)\,dt=$$
$$=\int_1^2 (2t^2-4t+4-4t+4)\,dt=$$
$$=\int_1^2 (2t^2-8t+8)\,dt=2\int_1^2 (t-2)^2\,dt=$$
$$=2\left[\tfrac{1}{3}(t-2)^3\right]_1^2=\frac{2}{3}.$$

Svar: Det utförda arbetet är $\frac{1}{3}$.

Uppgift 5.6 (Sid. 4)

Lösning

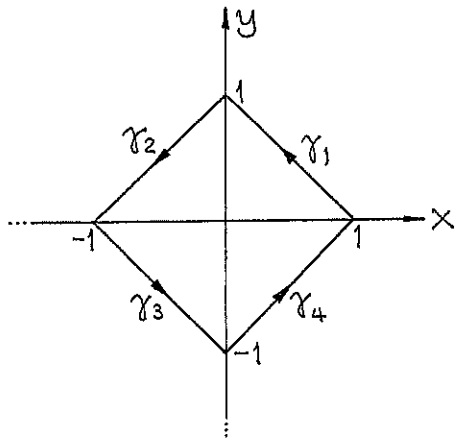


$F(x,y)=(-x,-y);\quad \gamma: r(t)=(3-2t,1+t),\ 0\leq t\leq 1.$

$$\int_\gamma F\cdot dr=\int_0^1 (2t-3,-t-1)\cdot(-2,1)\,dt=\int_0^1 (-5t+5)\,dt=\frac{5}{2}$$

Uppgift 5.7 (Sid. 4)

Lösning



$\gamma_1+\gamma_2+\gamma_3+\gamma_4=\gamma$

$A=(e^{x-y}, e^{x+y})$.

(1) $\gamma_1: r(t)=(1-t,t),\ 0\le t\le 1.$

$$\int_{\gamma_1} A\cdot dr=\int_0^1 (e^{1-2t}, e)\cdot(-1,1)\,dt=\int_0^1(-e^{1-2t}+e)\,dt=$$
$$=e\int_0^1(1-e^{-2t})\,dt=e\left[t+\tfrac{1}{2}e^{-2t}\right]_0^1=\tfrac{1}{2}(e+e^{-1});$$

(2) $\gamma_2: r(t)=(1-t,-t),\ 0\le t\le 1.$

$$\int_{\gamma_2} A\cdot dr=\int_0^1 (e, e^{1-2t})\cdot(-1,-1)\,dt=\int_0^1(-e-e^{1-2t})\,dt=$$
$$=\int_0^1(-e)(1+e^{-2t})\,dt=-e\left[t-\tfrac{1}{2}e^{-2t}\right]_0^1=$$
$$=(-e)\left(1-\tfrac{1}{2}(e^{-2}-1)\right)=-e\cdot\tfrac{1}{2}(3-e^{-2})=\tfrac{1}{2}(e^{-1}-3e);$$

(3) $\gamma_3: r(t)=(t-1,-t),\ 0\le t\le 1.$

$$\int_{\gamma_3} A\cdot dr=\int_0^1 (e^{2t-1}, e^{-1})\cdot(1,-1)\,dt=\int_0^1(e^{2t-1}-e^{-1})\,dt=$$
$$=e^{-1}\int_0^1(e^{2t}-1)\,dt=e^{-1}\left[\tfrac{1}{2}e^{2t}-t\right]_0^1=\tfrac{1}{2}(e-3e^{-1}).$$

(4) $\gamma_4: r(t)=(t,t-1),\ 0\le t\le 1.$

$$\int_{\gamma_4} A\cdot dr=\int_0^1 (e, e^{2t-1})\cdot(1,1)\,dt=\int_0^1(e+e^{2t-1})\,dt=$$
$$=\left[e\cdot t+\tfrac{1}{2}e^{2t-1}\right]_0^1=e+\tfrac{1}{2}(e-e^{-1})=\tfrac{1}{2}(3e-e^{-1}).$$

$$\int_{\gamma} A\cdot dr=\tfrac{1}{2}(e+e^{-1}+e^{-1}-3e+e-3e^{-1}+3e-e^{-1})=e-e^{-1}.$$

Uppgift 5.8 (Sid. 4)

Lösning

$\gamma: x^2+y^2=1$; $D=\{(x,y): x^2+y^2\le 1\}$.

Greens formel: $\oint_\gamma \omega=\iint_D d\omega=\iint_D(\partial_x Q-\partial_y P)\,dx\,dy$

$$\oint_\gamma A\cdot dr=\oint_\gamma(-y^3, x^3)\cdot(dx,dy)=3\iint_D(x^2+y^2)\,dx\,dy=$$
$$=\left[\begin{array}{l|l} x=r\cos\theta & 0\le r\le 1 \\ y=r\sin\theta & 0\le\theta\le 2\pi\end{array}\right]=3\int_0^1 r^3\,dr\int_0^{2\pi}d\theta=\tfrac{3}{4}\cdot 2\pi=\tfrac{3\pi}{2}.$$

Greens formel härleds/bevisas i teoriboken.

Anm. $\gamma: r(t)=(\cos t, \sin t)$, $0\le t\le 2\pi$;

$$\oint_\gamma A\cdot dr=\int_0^{2\pi}(-\sin^3 t,\cos^3 t)\cdot(-\sin t,\cos t)\,dt=$$

$$=\int_0^{2\pi}(\cos^4 t+\sin^4 t)\,dt=$$

$$=\int_0^{2\pi}((\cos^2 t)^2+(\sin^2 t)^2)\,dt=$$

$$=\int_0^{2\pi}\left(\left(\frac{1+\cos 2t}{2}\right)^2+\left(\frac{1-\cos 2t}{2}\right)^2\right)=$$

$$=\int_0^{2\pi}\frac{1}{2}(1+\cos^2 2t)\,dt=$$

$$=\frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2}+\frac{1}{2}\cos 4t\right)dt=\frac{3\pi}{2}.$$

Uppgift 5.9 (Sid. 5)

Lösning

a) $\gamma: r(t)=(\cos t,\sin t)$, $0\le t\le$ ..

$$\omega(\gamma)=(3\cos^2 t+4\sin t)(-\sin t)\,dt+$$

$$+(\sin^4 t+3\sin^2 t+\cos t)\cos t\,dt=$$

$$=(-3\cos^2 t\sin t-4\sin^2 t+\sin^4 t\cos t+$$

$$+3\sin^2 t\cos t+\cos^2 t)\,dt=$$

$$=(\underbrace{-3\cos^2 t\sin t}_{(\cos^3 t)'}-2+\underbrace{2\cos 2t}_{(\sin 2t)'}+\underbrace{\sin^4 t\cos t}_{(\frac{1}{5}\sin^5 t)'}+$$

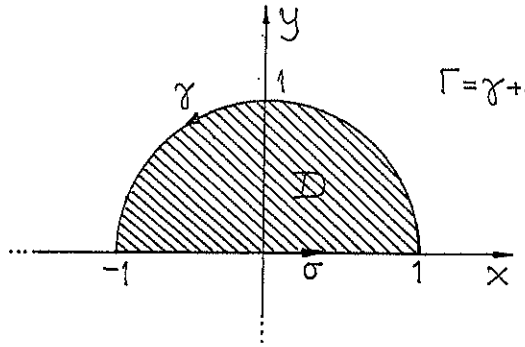
$$+\underbrace{3\sin^2 t\cos t}_{(\sin^3 t)'}+\frac{1}{2}+\underbrace{\frac{1}{2}\cos 2t}_{(\frac{1}{2}\sin 2t)'})\,dt=(\cos^3 t-\frac{3t}{2}+\sin 2t+$$

$$+\frac{1}{5}\sin^5 t+\sin^3 t+\frac{1}{4}\sin 2t)'\,dt;$$

$$\oint_\gamma A\cdot dr=\left[\cos^3 t-\frac{3t}{2}+\sin 2t+\frac{1}{5}\sin^5 t+\sin^3 t+\frac{\sin 2t}{4}\right]_0^\pi$$

$$=\underline{-2-\frac{3\pi}{2}}.$$

b) Jag stänger halvcirkeln med diametern utmed x-axeln.



$$\omega=(3x^2+4y)\,dx+(y^4+3y^2+x)\,dy \Rightarrow d\omega=-3\,dx\,dy$$

$$\Rightarrow \oint_\Gamma \omega=-3\iint_D dx\,dy=-\frac{3}{2}\pi \Leftrightarrow \int_\gamma \omega+\int_\sigma \omega=-3\pi/2$$

$$\Leftrightarrow \int_\gamma \omega=-\int_\sigma \omega-3\pi/2=-\int_{-1}^{1}3x^2\,dx-\frac{3\pi}{2}=-2-\frac{3\pi}{2}.$$

Uppgift 5.10 (Sid. 5)

Lösning: Parametrisering är att föredra!

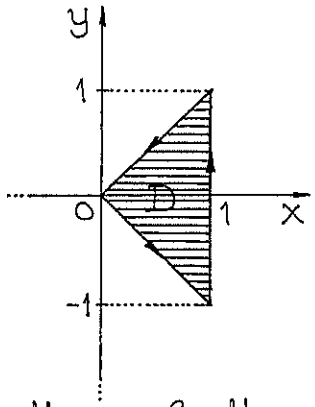
$\gamma: r(t)=(\cos t, \sin t),\ 0\le t\le 2\pi;\ A=\frac{(-y,x)}{x^2+y^2}$.

$$\oint_\gamma A\cdot dr=\int_0^{2\pi}(-\sin t,\cos t)\cdot(-\sin t,\cos t)\,dt=$$
$$=\int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt=\int_0^{2\pi}dt=2\pi.$$

Läs anmärkningen på sidan 340 i teoriboken (Persson & Böiers).

Uppgift 5.11 (Sid. 5)

Lösning



$$\omega=A\cdot dr=\frac{2xe^y}{(1+x^2)^2}dx-\frac{y^2+e^y}{1+x^2}dy\Rightarrow d\omega=\left(\partial_x\frac{-(y^2+e^y)}{1+x^2}-\right.$$
$$\left.-\partial_y\frac{2xe^y}{(1+x^2)^2}\right)dxdy=\left(\frac{2x(y^2+e^y)}{(1+x^2)^2}-\frac{2xe^y}{(1+x^2)^2}\right)dxdy=$$
$$=\frac{2xy^2}{(1+x^2)^2}dxdy;$$

$$\int_\gamma\omega=\iint_D d\omega=\int_0^1 dx\,\frac{2x}{(1+x^2)^2}\int_{-x}^{x}y^2dx=\frac{2}{3}\int_0^1\frac{2x^4}{(1+x^2)^2}dx=$$
$$=\frac{2}{3}\cdot\int_0^1\frac{2x}{(1+x^2)^2}\cdot x^3dx=\frac{2}{3}\left[-\frac{x^3}{x^2+1}\right]_0^1+2\int_0^1\frac{x^2}{x^2+1}dx=$$
$$=-\frac{1}{3}+2\int_0^1\left(1-\frac{1}{x^2+1}\right)dx=-\frac{1}{3}+2-2\operatorname{atn}1=\underline{\frac{5}{3}-\frac{\pi}{2}}.$$

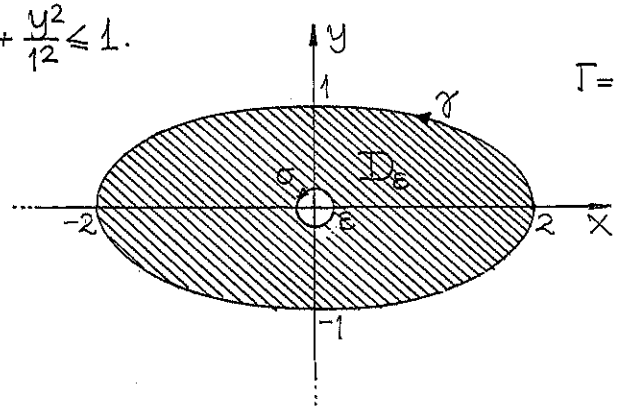
Uppgift 5.12 (Sid. 5)

Lösning

$\omega=\left(\frac{y}{x^2+y^2}-y\right)dx+\left(x-\frac{x}{x^2+y^2}\right)dy$; $\gamma: x^2+4y^2=4$.

(1) $x^2+4y^2=4\Leftrightarrow\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$, ellipskurva.

(2) $D:\ \frac{x^2}{2^2}+\frac{y^2}{1^2}\le 1$.



(3) $D_\varepsilon=\{(x,y):\ x^2+4y^2\le 4,\ x^2+y^2\ge\varepsilon^2\}$

(4) $\exists\ D_\varepsilon$ är fältet $A=\left(\frac{y}{x^2+y^2}-y,\frac{-x}{x^2+y^2}+x\right)$ $\mathcal{C}^1$ s.a.

Greens formel kan användas:

$$d\omega=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=2dxdy\Rightarrow\oint_\Gamma\omega=\underbrace{2(2\pi-\pi\varepsilon^2)}_{\mathcal{A}(D_\varepsilon)}$$

$= 4\pi - 2\pi\varepsilon^2 + \int_0^{2\pi} \left(\frac{\sin t}{\varepsilon} - \varepsilon\sin t, \varepsilon\cos t - \frac{\cos t}{\varepsilon}\right)\cdot(-\varepsilon\sin t, \varepsilon\cos t)\,dt = 4\pi - 2\pi\varepsilon^2 + \int_0^{2\pi} (-\sin^2 t - \cos^2 t)(1-\varepsilon^2)\,dt =$

$= 4\pi - 2\pi\varepsilon^2 - 2\pi(1-\varepsilon^2) = \underline{2\pi}$ (oberoende av ε).

Anm. Randen Γ ska vara positivt orienterad (det inre till vänster).

Uppgift 5.13 (Sid. 5)

Lösning

a) $\omega = -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy \Rightarrow d\omega = \left(\partial_x \frac{1}{2}x - \partial_y\left(-\frac{1}{2}y\right)\right)dx\,dy = dx\,dy \Rightarrow \oint_\gamma = \iint_D d\omega = \iint_D dx\,dy = \mathcal{A}(D)$.

b) $\gamma: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Leftrightarrow \underline{\gamma: r(t) = (a\cos t, b\sin t),\ 0 \le t \le 2\pi}$

$\mathcal{A}(D) = \frac{1}{2}\oint_\gamma -y\,dx + x\,dy = \frac{1}{2}ab\int_0^{2\pi} dt = \frac{ab}{2}2\pi = \underline{\pi ab}$.

Uppgift 5.14 (Sid. 5)

Lösning

$\underline{\gamma: r(t) = (\cos^3 t, \sin^3 t),\ 0 \le t \le 2\pi}$.

$\mathcal{A}(D) = \frac{1}{2}\oint_\gamma -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} 3\cos^2 t \sin^2 t\,dt =$

$= \frac{3}{2}\int_0^{2\pi} \left(\frac{1}{2}\sin 2t\right)^2 dt = \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,dt =$

$= \frac{3}{8}\int_0^{2\pi} \frac{1}{2}(1 - \cos 4t)\,dt =$

$= \frac{3}{8}\cdot\frac{1}{2}\cdot 2\pi = \underline{\frac{3\pi}{8}}$.

Uppgift 5.15 (Sid. 5)

Lösning

Låt σ vara sträckan mellan $(-1,0)$ och $(1,0)$.

$\Gamma = \gamma + \sigma \Rightarrow \oint_\Gamma \omega = \iint_D (\partial_x(6x - x^3) - \partial_y(y^3 - 6y))\,dx\,dy =$

$= \iint_D (6 - 3x^2 - 3y^2 + 6)\,dx\,dy = 3\iint_D (4 - x^2 - y^2)\,dx\,dy$

$\Rightarrow \underline{4 - x^2 - y^2 \ge 0}$, för maximalt värde av $\int_\gamma \omega$;

$\int_\gamma \omega + \int_\sigma \omega = 3\iint_D (4 - x^2 - y^2)\,dx\,dy = /\text{polärt}/ =$

$= 3\int_0^2 (4 - r^2)\,r\,dr \int_0^\pi d\theta = 3 \times \pi\left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 12\pi$

$\Leftrightarrow \int_\gamma \omega = 12\pi - \int_\sigma \omega = \underline{12\pi}$.

Svar: Kurvan γ som maximerar $\int_\gamma \omega$ är

$\gamma: r(t) = (2\cos t, 2\sin t),\ 0 \le t \le \pi$.

Det största värdet blir i detta fall 12π.

Uppgift 5.16 (Sid. 5)

Lösning

$$A = (P(x,y),\ Q(x,y)) = \left(\frac{y^3}{3} - 2xy + 4y,\ -\frac{x^3}{3} + 4xy\right) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} P = \frac{1}{3}y^3 - 2xy + 4y \Rightarrow \partial_y P = y^2 - 2x + 4 \\ Q = -\frac{1}{3}x^3 + 4xy \Rightarrow \partial_x Q = -x^2 + 4y \end{cases} \Rightarrow \partial_x Q - \partial_y P =$$

$$= -x^2 + 4y - y^2 + 2x - 4 = -(x^2 + y^2 - 2x - 4y + 4) =$$

$$= -((x-1)^2 + (y-2)^2 - 1) = \underline{1 - (x-1)^2 - (y-2)^2};$$

$$\oint_\gamma A \cdot dr = \iint_D (1 - (x-1)^2 - (y-2)^2)\, dx dy,\ \underline{D = int(\gamma)}$$

$\oint_\gamma A \cdot dr$ blir maximalt om och endast om integranden är icke-negativ, dvs om

$$1 - (x-1)^2 - (y-2)^2 \geqslant 0 \Leftrightarrow \underline{D: (x-1)^2 + (y-2)^2 \leq 1}.$$

Int(γ) = det inre av γ, dvs området D.

$$\oint_\gamma A \cdot dr = \iint_D (1 - (x-1)^2 - (y-2)^2)\, dx dy = \begin{bmatrix} x = 1 + r\cos\theta \\ y = 2 + r\sin\theta \end{bmatrix} =$$

$$= \int_0^1 (1 - r^2) r\, dr \int_0^{2\pi} d\theta = \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 \cdot 2\pi = \frac{\pi}{2}.$$

Svar: $\gamma: r(t) = (1 + \cos t,\ 2 + \sin t),\ 0 \leq t \leq 2\pi;\ \int_\gamma \omega = \frac{\pi}{2}$.

6. Ytintegraler, flödesintegrale, Gauss's sats

Uppgift 6.1 (Sid. 5)

Lösning

$\underline{S: r(u,v) = (\cos u, \sin u, v);\ 0 \leq u \leq 2\pi,\ 0 \leq v \leq 1.}$

$$\begin{cases} r'_u = (-\sin u, \cos u, 0) \\ r'_v = (0, 0, 1) \end{cases} \Rightarrow r'_u \times r'_v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\cos u, \sin u, 0) \Rightarrow dS = |r'_u \times r'_v| du dv = du dv$$

$$\Rightarrow \iint_S x^2 dS = \int_0^{2\pi} \cos^2 u\, du \cdot \int_0^1 dv = \frac{1}{2} \cdot 2\pi = \underline{\pi}.$$

Uppgift 6.2 (Sid. 5)

Lösning

$\underline{S: r(u,v) = (u\cos v, u\sin v, u^2);\ 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi.}$

$$\begin{cases} r'_u = (\cos v, \sin v, 2u) \\ r'_v = (-u\sin v, u\cos v, 0) \end{cases} \Rightarrow r'_u \times r'_v = (-2u^2\cos v)\hat{x} +$$

$$+ (-2u^2\sin v)\hat{y} + u\hat{z} \Rightarrow |r'_u \times r'_v| = \sqrt{4u^2+1}\, u \Rightarrow$$

$dS = |r'_u \times r'_v| du dv = \sqrt{4u^2+1}\, u\, du dv$; forts

$$\iint_S \frac{1}{1+4z}\,dS = \int_0^1 \frac{\sqrt{1+4u^2}}{1+4u^2}\,u\,du \int_0^{2\pi} dv = 2\pi \int_0^1 \frac{u}{\sqrt{4u^2+1}}\,du =$$

$$= /\,4u^2+1 = t \Rightarrow u\,du = \frac{1}{8}dt\,/ = \frac{2\pi}{8}\int_1^5 \frac{dt}{\sqrt{t}} = \frac{\pi}{2}\left[\sqrt{t}\right]_1^5 =$$

$$= \frac{\pi}{2}\cdot(\sqrt{5}-1).$$

Uppgift 6.3 (Sid. 5)

Lösning

$S: r(u,v) = (u\cos v, u\sin v, u^2);\ 0\le u\le 1,\ 0\le v\le 2\pi.$

$$r'_u \times r'_v = (-2u^2\cos v, -2u^2\sin v, u) \Rightarrow \hat{n}\,dS = (!) =$$

$$= -(r'_u \times r'_v)\,du\,dv = (2u^2\cos v, 2u^2\sin v, -u)\,du\,dv$$

$$\Rightarrow A(r)\cdot\hat{n}\,dS = 2u^3(\cos^2 v + \sin v\cos v + u\cos v)\,du\,dv$$

$$\Rightarrow \iint_S A\cdot\hat{n}\,dS = \int_0^1 u^3\,du \int_0^{2\pi} 2\cos^2 v\,dv = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}.$$

Anm. $\int_0^{2\pi} \sin x\cos x\,dx = \int_0^{2\pi} \sin x\,dx = \int_0^{2\pi} \cos x\,dx = 0.$

I (!) underförstås villkoret $\hat{n}\cdot\hat{z} < 0$.

Uppgift 6.4 (Sid. 5)

Lösning

$S: r(u,v) = (u, v, 1-u-v);\ D: u\ge 0,\ v\ge 0,\ u+v\le 1.$

$$r'_u \times r'_v = (1,0,-1)\times(0,1,-1) = (1,1,1) \Rightarrow dw = F\cdot dS =$$

$$= F\cdot\hat{n}\,dS = (u+1, -2v-1, 1-u-v)\cdot(1,1,1)\,du\,dv =$$

$$= (u+1-2v-1+1-u-v)\,du\,dv = (1-3v)\,du\,dv;$$

$$\iint_D F\cdot\hat{n}\,dS = \int_0^1 du \int_0^{1-u} (1-3v)\,dv =$$

$$= \int_0^1 \left(\left[v - \frac{3}{2}v^2\right]_{v=0}^{v=1-u}\right) du =$$

$$= \int_0^1 \left(1-u-\frac{3}{2}(1-u)^2\right) du =$$

$$= \left[\frac{1}{2}(1-u)^3 - \frac{1}{2}(1-u)^2\right]_0^1 = \underline{0}.$$

Uppgift 6.5 (Sid. 6)

Lösning: Jag tillämpar Gauss' sats.

$$A(x,y,z) = (x^3y^2z^2,\ x^2y^3z^2,\ x^2y^2z^3);\quad :\begin{cases} |x|\le 1 \\ |y|\le 1; \\ |z|\le 1 \end{cases}$$

$$\operatorname{div} A = \partial_x(x^3y^2z^2) + \partial_y(x^2y^3z^2) + \partial_z(x^2y^2z^3)$$

$$= 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2 = 9x^2y^2dz$$

$$\oiint_{\partial V} A\cdot\hat{n}\,dS = \iiint_V \operatorname{div} A\,dV = \iiint_V 9x^2y^2z^2\,dx\,dy\,dz =$$

$$= 9\int_{-1}^1 x^2dx \int_{-1}^1 y^2dy \int_{-1}^1 z^2dz = 9\left(\int_{-1}^1 x^2dx\right)^3 =$$

$$= 9\cdot\left(\frac{2}{3}\right)^3 = 9\cdot\frac{8}{27} = \frac{8}{3}.$$

Uppgift 6.6 (Sid. 6)

Lösning

$$\underline{A(x,y,z)=(0,x+z,0)}.$$

$\underline{S:\ r(u,v)=(\cos u+v,\ \cos v,\ v+1);\ 0\le u\le\frac{\pi}{2},\ 0\le v\le 1.}$

$$A\cdot\hat{n}\,dS = A(r(u,v))\cdot(r_u'\times r_v')\,du\,dv =$$

$$=(0,\cos u+2v+1,0)\cdot(0,\sin u,\sin u\sin v)\,du\,dv =$$

$$=\sin u(\cos u+2v+1)\,du\,dv =$$

$$=(\sin u\cos u+(2v+1)\sin u)\,du\,dv\ ;$$

$$\iint_S A\cdot\hat{n}\,dS=\int_0^1 dv\int_0^{\pi/2}\left(\tfrac{1}{2}\sin 2u+(2v+1)\sin u\right)du =$$

$$=\int_0^1\left(\left[-\tfrac{1}{4}\cos 2u-(2v+1)\cos u\right]_{u=0}^{\pi/2}\right)dv =$$

$$=\int_0^1\left(\tfrac{1}{2}+2v+1\right)dv=\int_0^1\left(\tfrac{3}{2}+2v\right)dv=1+\tfrac{3}{2}=\tfrac{5}{2}.$$

Uppgift 6.7 (Sid. 6)

Lösning

$$\underline{A=(2x,-z,y)}.$$

$\underline{S:\ r(u,v)=(u,v\cos u,v\sin u);\ 0\le u\le\pi,\ 0\le v\le 1.}$

$$r_u'\times r_v'=(1,-v\sin u,v\cos u)\times(0,\cos u,\sin u)=$$

$$=(-v,\sin u,\cos u)\Rightarrow \underline{A\cdot\hat{n}\,dS}=A\cdot(-(r_u'\times r_v'))\,du\,dv=$$

$$=(2u,-v\sin u,v\cos u)\cdot(v,-\sin u,-\cos u)\,du\,dv=$$

$$=(2uv-v)\,du\,dv=\underline{(2u-1)v\,du\,dv};\ 0\le u\le\pi,\ 0\le v\le 1.$$

$$\iint_S A\cdot\hat{n}\,dS=\int_0^{\pi}(2u-1)\,du\int_0^1 v\,dv=\frac{\pi^2-\pi}{2}.$$

Uppgift 6.8 (Sid. 6)

Lösning

$$\underline{\mathbb{F}(x,y,z)=(x+z,y+z,z^2)}.$$

$S:\ r(u,v)=(\sqrt{2}\cos u,\sqrt{2}\sin u,v);\ 0\le u\le 2\pi,\ 0\le v\le 1$

$$r_u'\times r_v'=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\ -\sqrt{2}\sin u&\sqrt{2}\cos u&0\\ 0&0&1\end{vmatrix}=\sqrt{2}(\cos u,\sin u,0)$$

$$\Rightarrow \mathbb{F}\cdot\hat{n}\,dS=\mathbb{F}(r(u,v))\cdot(r_u'\times r_v')\,du\,dv=$$

$$=2(\cos^2 u+\cos u\sin u+2\sin^2 u)\,du\,dv=$$

$$=(2+2\cos u\sin u)\,du\,dv=$$

$$=(2+\sin 2u)\,du\,dv\ \Rightarrow$$

$$\iint_S\mathbb{F}\cdot\hat{n}\,dS=\int_0^{2\pi}(2+\sin 2u)\,du\int_0^1 dv=$$

$$=\left[2u-\tfrac{1}{2}\cos 2u\right]_0^{2\pi}\cdot\left[v\right]_0^1=\underline{4\pi}.$$

Uppgift 6.9 (Sid. 6)

Lösning

$$\mathbb{A}(x,y,z) = e^{x^2+y^2}(x,y,z)$$

$S: \mathbf{r}(u,v) = (u,v,1-u^2-v^2)$, $\mathbb{D}: u^2+v^2 \le 1$.

$$\mathbb{A}\cdot\hat{n}\,dS = e^{u^2+v^2}(u,v,1-u^2-v^2)\cdot(2u,2v,1)\,du\,dv =$$

$$= e^{u^2+v^2}(2u^2+2v^2+1-u^2-v^2)\,du\,dv =$$

$$= e^{u^2+v^2}(u^2+v^2+1)\,du\,dv;$$

$$\iint_S \mathbb{A}\cdot\hat{n}\,dS = \iint_{\mathbb{D}} e^{u^2+v^2}(u^2+v^2+1)\,du\,dv = \begin{bmatrix} u = r\cos\theta \\ v = r\sin\theta \end{bmatrix} =$$

$$= \int_0^1 e^{r^2}(r^2+1)r\,dr \int_0^{2\pi} d\theta = \pi\int_0^1 (1+r^2)e^{r^2}2r\,dr =$$

$$= /t = r^2/ = \pi\int_0^1 (t+1)e^t\,dt = \pi[te^t]_0^1 = \pi e.$$

Uppgift 6.10 (Sid. 6)

Lösning

$x^2+y^2-z^2+4z = 4 \Leftrightarrow z^2-4z+4 = x^2+y^2 \Leftrightarrow (z-2)^2 =$

$= x^2+y^2 \Leftrightarrow z-2 = \pm\sqrt{x^2+y^2} \Leftrightarrow z = 2-\sqrt{x^2+y^2}$ $(0\le z\le 2)$.

$S: \mathbf{r}(u,v) = (u\cos v, u\sin v, 2-u)$, $0\le u\le 2$, $0\le v\le 2\pi$;

$$\mathbf{r}'_u \times \mathbf{r}'_v = (\cos v, \sin v, -1)\times(-u\sin v, u\cos v, 0) =$$

$$= u(\cos v, \sin v, 1).$$

$$\mathbb{A}(x,y,z) = (x^2+y^2, y^2+z^2, x^2+z^2);$$

$$\mathbb{A}(\mathbf{r}(u,v)) = (u^2, u^2\sin^2 v+(2-u)^2, u^2\cos^2 v+(2-u)^2);$$

$$\mathbb{A}\cdot\hat{n}\,dS = \mathbb{A}(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u\times\mathbf{r}'_v)\,du\,dv =$$

$$= (u^3\cos v + u^3\sin^3 v + u(2-u)^2\sin v + u^3\cos^2 v + u(2-u)^2)\,du\,dv$$

$$\iint_S \mathbb{A}\cdot\hat{n}\,dS = \int_0^2 u^3\,du\int_0^{2\pi}\cos v\,dv + \int_0^2 u^3\,du\int_0^{2\pi}\sin^3 v\,dv +$$

$$+ \int_0^2 u(u-2)^2\,du\int_0^{2\pi}\sin v\,dv +$$

$$+ \int_0^2 u^3\,du\cdot\int_0^{2\pi}\cos^2 v\,dv + \int_0^2 u(2-u)^2\,du\int_0^{2\pi} dv$$

$$= 0+0+0+\pi\int_0^2 u^3\,du + 2\pi\int_0^2 (u^3-4u^2+4u)\,du$$

$$= 4\pi + \frac{8\pi}{3} = \frac{20\pi}{3}.$$

Anm. $\int_0^{2\pi}\cos x\,dx = \int_0^{2\pi}\sin x\,dx = 0.$

$\int_0^{2\pi}\cos^2 x\,dx = \int_0^{2\pi}\sin^2 x\,dx = \pi.$

$\int_0^{2\pi}\cos^3 x\,dx = \int_0^{2\pi}\sin^3 x\,dx = 0.$

Detta reducerar räkningarna avsevärt.

Uppgift 6.11 (Sid. 6)

Lösning

$S: \mathbf{r}(u,v) = (\sin u\cos v, \sin u\sin v, \cos u)$, $\begin{cases} 0\le u\le \pi \\ 0\le v\le 2\pi \end{cases}$;

$A(x,y,z) = xyz\,(1,1,1)$;

$$A\cdot\hat{n}\,dS = A(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u\times\mathbf{r}'_v)\,du\,dv =$$

$$= (\sin^3 u\cos u\cdot\cos^2 v\sin v + \sin^3 u\cos u\sin^2 v\cos v +$$

$$+\sin^2 u\cos^2 u\sin v\cos v)\,du\,dv;$$

$$\iint_S A\cdot\hat{n}\,dS = \int_0^{\pi}\sin^3 u\cos u\,du\int_0^{2\pi}\cos^2 v\sin v\,dv +$$

$$+\int_0^{\pi}\sin^3 u\cos u\,du\int_0^{2\pi}\sin^2 v\cos v\,dv +$$

$$+\int_0^{\pi}\sin^2 u\cos^2 u\,du\int_0^{\pi}\sin v\cos v\,dv = 0.$$

Uppgift 6.12 (Sid. 6)

Lösning

$S: \mathbf{r}(u,v) = (u\cos v, u\sin v, u)$; $1\le u\le 2$, $0\le v\le 2\pi$

$\hat{n}\,dS = (\mathbf{r}'_u\times\mathbf{r}'_v)\,du\,dv = (-u\cos v, -u\sin v, u)\,du\,dv$

$A(\mathbf{r}(u,v)) = (u^2\cos^2 v,\ u^2\sin^2 v,\ u^2)$; forts

$$A\cdot\hat{n}\,dS = (-u^3\cos^3 v - u^3\sin^3 v + u^3)\,du\,dv$$

$$\oiint_S A\cdot\hat{n}\,dS = \int_1^2 u^3\,du\int_0^{2\pi}(-\cos^3 v - \sin^3 v + 1)\,dv =$$

$$= \int_1^2 u^3\,du\cdot 2\pi = \frac{2^4-1}{4}\cdot 2\pi = \underline{\underline{\frac{15\pi}{2}}}.$$

Uppgift 6.13 (Sid. 6)

Lösning

$\underline{A(x,y,z) = (xy^2z,\ x^2yz,\ xyz^2)}$; $D: \begin{cases} x^2+y^2\le 1 \\ z=0 \end{cases}$.

Gauss' sats kan tillämpas.

$$\operatorname{div} A = \partial_x(xy^2z) + \partial_y(x^2yz) + \partial_z(xyz^2) =$$

$$= y^2z + x^2z + 2xyz;$$

$$\oiint_{\partial V} A\cdot\hat{n}\,dS = \iiint_V (y^2z + x^2z + 2xyz)\,dx\,dy\,dz =$$

$$= \iiint_V (x^2+y^2)z\,dx\,dy\,dz + 2\iiint_V xyz\,dV =$$

$$= \iint_D\left(\int_0^1 z\,dz\right)(x^2+y^2)\,dx\,dy = \begin{bmatrix} x = r\cos v \\ y = r\sin v \end{bmatrix} =$$

$$= \frac{1}{2}\int_0^1 r^3\,dr\int_0^{2\pi} d\theta = \underline{\underline{\frac{\pi}{4}}}.$$

Anm. $\iiint_V xyz\,dV = \int_0^1 z\,dz\int_{-1}^1 dx\times\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\,dy = 0.$

Uppgift 6.14 (Sid. 6)

Lösning

Jag kommer att använda Gauss' sats.

$\mathbf{F}(x,y,z) = (x^2yz, xy^2z, xyz^2) \Rightarrow \operatorname{div}\mathbf{F} = 6xyz$;

V: $x^2+y^2+z^2 \le 1$; $x,y,z \ge 0$; D: $x^2+y^2 \le 1$, $x,y \ge 0$.

D är V:s projektion i xy-planet.

$$\oiint_{\partial V} \mathbf{F}\cdot\hat{n}\,dS = 6\iiint_V xyz\,dxdydz =$$

$$= 3\iint_D \left(\int_0^{\sqrt{1-x^2-y^2}} 2z\,dz\right) xy\,dxdy =$$

$$= 3\iint_D (1-x^2-y^2)xy\,dxdy =$$

$$= \left[\begin{array}{l|l} x = r\cos\theta & 0 \le r \le 1 \\ y = r\sin\theta & 0 \le \theta \le \pi/2 \end{array}\right] =$$

$$= 3\int_0^1 (1-r^2)r^3\,dr \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta =$$

$$= 3\int_0^1 (r^3-r^5)\,dr \cdot \frac{1}{2}\int_0^{\pi/2} \sin 2\theta\,d\theta =$$

$$= \frac{3}{2}\left[\frac{1}{4}r^4 - \frac{1}{6}r^6\right]_0^1 \cdot \left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} =$$

$$= \frac{3}{2}\cdot\frac{1}{12}\cdot 1 = \underline{\frac{1}{8}}.$$

Uppgift 6.15 (Sid. 6)

Lösning

$x^2+y^2+z^2 = 3+4x+2y+2z \Leftrightarrow (x^2-4x+4)+(y^2-2y+1)+$

$+(z^2-2z+1) = 3+1+4+1 = 9 \Leftrightarrow \underline{(x-2)^2+(y-1)^2+(z-1)^2 = 3^2}$.

Jag kommer att använda Gauss' sats:

$\mathbf{A}(x,y,z) = (y^2, 2xy, 0) \Rightarrow \operatorname{div}\mathbf{A} = 2x$; $S = \partial V$.

$$\oiint_S \mathbf{A}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\mathbf{A}\,dV = 2\iiint_V x\,dxdydz;$$

$$\begin{cases} x = 2 + r\sin u\cos v \\ y = 1 + r\sin u\sin v \\ z = 1 + r\cos v \end{cases} \Rightarrow dxdydz \to \left|\frac{\partial(x,y,z)}{\partial(r,u,v)}\right| dr\,du\,dv$$

$dV = r^2\sin v\,dr\,du\,dv$; $\Delta: \begin{cases} 0 \le r \le 3 \\ 0 \le u \le 2\pi \\ 0 \le v \le \pi \end{cases}$.

$$\oiint_S \mathbf{A}\cdot\hat{n}\,dS = \iiint_\Delta 2(2+r\sin u\cos v)r^2\sin v\,dr\,du\,dv =$$

$$= 2\int_0^3 2r^2\,dr\int_0^{2\pi} du\int_0^{\pi} \sin v\,dv +$$

$$+\int_0^3 r^3\,dr\int_0^{2\pi} \sin u\,du\int_0^{\pi} \sin 2v\,dv =$$

$$= 2\left[\frac{2}{3}r^3\right]_0^3 \cdot 2\pi\cdot 2 = 2^2\cdot 9\cdot 2\pi\cdot 2 = \underline{144\pi}.$$

Se f.ö. Anm. på sidan 293 i Persson/Böiers.

Uppgift 6.16 (Sid. 6)

Lösning

Jag kommer att använda Gauss' sats:

$$\mathbf{A}(x,y,z)=(x^3,y^3,z^3)\Rightarrow \operatorname{div}\mathbf{A}=3(x^2+y^2+z^2);$$

$$K:\ \sqrt{x^2+y^2}\le z\le 1;\quad D:\ x^2+y^2\le 1,\ z=0.$$

$$\oiint_c \mathbf{A}\cdot\hat{n}\,dS=\iiint_K \operatorname{div}\mathbf{A}\,dV=3\iiint_K (x^2+y^2+z^2)\,dx\,dy\,dz=$$

$$=\iint_D\left(3\int_{\sqrt{x^2+y^2}}^{1}(x^2+y^2+z^2)\,dz\right)dx\,dy=$$

$$=\iint_D\left(\left[3(x^2+y^2)z+z^3\right]_{z=\sqrt{x^2+y^2}}^{1}\right)dx\,dy=$$

$$=\iint_D\left(3(x^2+y^2)+1-4(x^2+y^2)^{3/2}\right)dx\,dy=$$

$$=\left[\begin{array}{l|l} x=r\cos\theta & 0\le r\le 1 \\ y=r\sin\theta & 0\le\theta\le 2\pi \end{array}\right| dx\,dy\longrightarrow r\,dr\,d\theta\Big]=$$

$$=\int_0^1(3r^2+1-4r^3)r\,dr\int_0^{2\pi}d\theta=$$

$$=\int_0^1(3r^3+r-4r^4)\,dr\cdot 2\pi$$

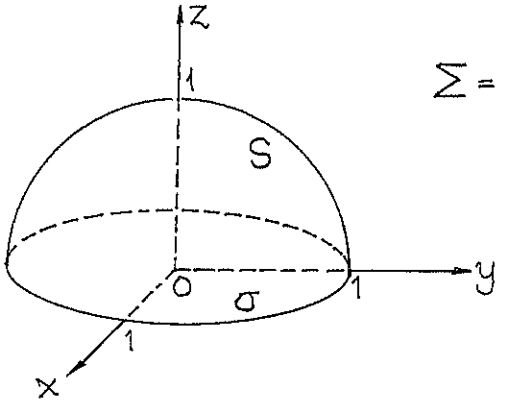
$$=2\pi\left[\frac{3}{4}r^4+\frac{1}{2}r^2-\frac{4}{5}r^5\right]_0^1=2\pi\cdot\frac{9}{20}=\frac{9\pi}{10}.$$

Gauss' sats tillämpas på slutna ytor

Uppgift 6.17 (Sid. 6)

Lösning

Jag stänger halvsfären med enhetsskivan $x^2+y^2\le 1$, $z=0$, och tillämpar Gauss' sats:



$$\mathbf{A}(x,y,z)=(xzy^2,\ x^2yz,\ x^2+y^2)\Rightarrow \operatorname{div}\mathbf{A}=(x^2+y^2)z;$$

$$K:\ x^2+y^2+z^2=1,\ z\ge 0;\quad D:\ x^2+y^2\le 1,\ z=0.$$

$$\Sigma=\partial K=S+D;\quad S:\ x^2+y^2+z^2=1,\ z\ge 0.$$

$$\oiint_\Sigma \mathbf{A}\cdot\hat{n}\,dS=\iiint_K \operatorname{div}\mathbf{A}\,dx\,dy\,dz=$$

$$=\iiint_K (zy^2+x^2z)\,dx\,dy\,dz=$$

$$=\iint_D (x^2+y^2)\left(\int_0^{\sqrt{1-x^2-y^2}} z\,dz\right)dx\,dy=$$

$$=\frac{1}{2}\iint_D (x^2+y^2)\cdot\left[z^2\right]_{z=0}^{z=\sqrt{1-x^2-y^2}}dx\,dy=$$

$$= \iint_D \frac{1}{2}(x^2+y^2)(1-x^2-y^2)\,dxdy =$$

$$= \left[\begin{array}{l|l} x = r\cos\theta & 0 \le r \le 1 \\ y = r\sin\theta & 0 \le \theta \le 2\pi \end{array} \;\middle|\; dxdy \longrightarrow rdrd\theta\right] =$$

$$= \frac{1}{2}\int_0^1 (r^2-r^4)\cdot r\,dr \int_0^{2\pi} d\theta =$$

$$= \pi \int_0^1 (r^3-r^5)\,dr = \frac{\pi}{12} \Leftrightarrow$$

$$\iint_S \mathbf{A}\cdot\hat{n}\,dS = \frac{\pi}{12} - \iint_D \mathbf{A}\cdot\hat{n}\,dS = \frac{\pi}{12} + \iint (x^2+y^2)\,dxdy =$$

$$= \frac{\pi}{12} + \int_0^1 r^3 dr \int_0^{2\pi} d\theta = \frac{\pi}{12} + \frac{\pi}{2} = \underline{\frac{7\pi}{12}}.$$

Anm. "Lockets" utåtriktade normal är $\hat{n} = -\hat{z}$

så att $\mathbf{A}\cdot\hat{n}\,dS = (x^2+y^2)\,dS = (x^2+y^2)\,dxdy$.

Uppgift 6.18 (Sid. 6)

Lösning

$$\underline{\mathbf{A}(x,y,z) = (x,y,z)}$$

a) $S: \mathbf{r}(u,v) = (R\cos u, R\sin u, v)$, $0 \le u \le 2\pi$, $-a \le v \le a$;

$$\mathbf{A}\cdot\hat{n}\,dS \longrightarrow \mathbf{A}(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u \times \mathbf{r}'_v)\,dudv =$$

$$= (R\cos u, R\sin u, v)\cdot(R\cos u, R\sin u, 0)\,dudv =$$

$$= R^2\,dudv \Rightarrow \iint_S \mathbf{A}\cdot\hat{n}\,dS = R^2\cdot 2\pi R\cdot 2a = \underline{4\pi a R^3}.$$

b) $x^2+y^2 = 4z^2 \Leftrightarrow 2z = \sqrt{x^2+y^2} \Leftrightarrow z = \frac{1}{2}\sqrt{x^2+y^2}$, $0 \le z \le 1$

$\underline{S: \mathbf{r}(u,v) = (u\cos v, u\sin v, \frac{1}{2}u);\ 0 \le u \le 1,\ 0 \le v \le 2\pi}$.

$$\mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos v & \sin v & 1/2 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-\tfrac{1}{2}u\cos v, -\tfrac{1}{2}u\sin v, u);$$

$$\mathbf{A}\cdot\hat{n}\,dS = \mathbf{A}(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u \times \mathbf{r}'_v)\,dudv =$$

$$= (u\cos v, u\sin v, \tfrac{u}{2})\cdot(-\tfrac{u}{2}\cos v, -\tfrac{u}{2}\sin v, u)\,dudv$$

$$= (-\tfrac{u}{2} - \tfrac{u}{2})\,dudv = 0 \Rightarrow \iint_S \mathbf{A}\cdot\hat{n}\,dS = \underline{0}.$$

Uppgift 6.19 (Sid. 7)

Lösning

a) $S: \mathbf{r}(u,v) = (\cos v, \sin v, u)$, $0 \le u \le 1$, $0 \le v \le 2\pi$

$$\mathbf{A}\cdot\hat{n}\,dS \longrightarrow \mathbf{A}(\mathbf{r}(u,v))\cdot(-\mathbf{r}'_u \times \mathbf{r}'_v)\,dudv =$$

$$= (\cos v, \sin v, u)\cdot(+\cos v, -\sin v, 0)\,dudv =$$

$$= -dudv;$$

$$\iint_S \mathbf{A}\cdot\hat{n}\,dS = -\iint_S dudv = -\mathcal{A}(S) = \underline{-2\pi}.$$

b) $x^2+y^2=z+1$; $0\le z\le 1 \Leftrightarrow 1\le x^2+y^2\le 2 \Leftrightarrow 1\le\sqrt{x^2+y^2}\le\sqrt{2}$.

$S: r(u,v)=(u\cos v, u\sin v, u^2-1)$, $1\le u\le\sqrt{2}$, $0\le v\le 2\pi$.

$r_u'\times r_v' = (\cos v, \sin v, 2u)\times(-u\sin v, u\cos v, 0) =$

$= u(-2u\cos v, -2u\sin v, 1)$

$A\cdot\hat{n}\,dS \longrightarrow A(r(u,v))\cdot(r_u'\times r_v')\,du\,dv =$

$= \frac{1}{u}(\cos v, \sin v, 0)\cdot u(-2u\cos v, -2u\sin v, 1)\,du\,dv =$

$= \underline{-2u\,du\,dv}$;

$$\iint_S A\cdot\hat{n}\,dS = -\int_1^{\sqrt{2}} 2u\,du\int_0^{2\pi} dv = -\left[u^2\right]_1^{\sqrt{2}}\cdot 2\pi = \underline{-2\pi}.$$

Uppgift 6.20 (Sid. 7)

Lösning

Jag kommer att tillämpa Gauss' sats.

$F(x,y,z)=(x,-y^2,x^2+z^2-1) \Rightarrow \operatorname{div} F = 1-2y+2z$

$S: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; $\quad V=\operatorname{int}(S): \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.

$$\oiint_S F(x,y,z)\cdot\hat{n}\,dS = \iiint_V (1-2y+2z)\,dx\,dy\,dz =$$

$$= \iiint_V dx\,dy\,dz - 2\iiint_V y\,dx\,dy\,dz + 2\iiint_V z\,dx\,dy\,dz =$$

$$= \left[\begin{array}{l|l} x=ar\sin u\cos v & \Delta: 0\le r\le 1,\ 0\le u\le\pi,\ 0\le v\le 2\pi \\ y=br\sin u\sin v & \\ z=cr\cos u & dx\,dy\,dz \to abcr^2\sin u\,dr\,du\,dv \end{array}\right]$$

$$= abc\int_0^1 r^2dr\cdot\int_0^\pi \sin u\,du\int_0^{2\pi} dv -$$

$$-2ab^2c\int_0^1 r^3dr\int_0^\pi \sin^2 u\,du\int_0^{2\pi}\sin v\,dv +$$

$$+2abc^2\int_0^1 r^3dr\int_0^\pi \sin u\cos u\,du\int_0^{2\pi} dv =$$

$$= abc\left[\tfrac{1}{3}r^3\right]_0^1\cdot\left[-\cos u\right]_0^\pi\cdot\left[v\right]_0^{2\pi} - 2ab^2c\cdot\frac{1}{4}\cdot\frac{\pi}{2}\cdot 0 +$$

$$+2abc^2\cdot\frac{1}{4}\cdot 0\cdot 2\pi = \underline{\frac{4\pi}{3}abc}.$$

Anm. $V: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ speglar i koordinatplanen (dvs xy-, yz- och xz-planen) så att

$$\underline{\iiint_V x\,dV = \iiint_V y\,dV = \iiint_V z\,dV = 0.}$$

Uppgift 6.21 (Sid. 7)

Lösning

a) $S: r(u,v)=R(\sin u\cos v, \sin u\sin v, \cos u)$;

$\Delta: 0\le u\le\pi,\ 0\le v\le 2\pi$;

$r_u'\times r_v' = R^2(\sin u\cos v, \sin u\sin v, \cos u)\sin u$;

$$\mathbf{F}(x,y,z)=(x^2,2y,z).$$

$$\mathbf{F}(\mathbf{r}(u,v))=(R^2\sin^2u\cos^2v,\ 2R\sin u\sin v,\ R\cos u)$$

$$\omega=\mathbf{F}\cdot\hat{n}\,dS=\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u\times\mathbf{r}'_v)\,du\,dv=$$

$$=(R^2\sin^3u\cos^3v+2R\sin^2u\sin^2v+R\cos^2u)\cdot dS=$$

$$=R^3(R\sin^4u\cos^3v+2\sin^3u\sin^2v+\cos^2u\sin u)\,du\,dv;$$

$$\oint\!\!\!\oint_S\mathbf{F}\cdot\hat{n}\,dS=\iint_\Delta\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u\times\mathbf{r}'_v)\,du\,dv=$$

$$=R^4\int_0^\pi\sin^4u\,du\int_0^{2\pi}\cos^3v\,dv+$$

$$+2R^3\int_0^\pi\sin^3u\,du\int_0^{2\pi}\sin^2v\,dv+$$

$$+R^3\int_0^\pi\cos^2u\sin u\,du\int_0^{2\pi}dv=$$

$$=2R^3\int_0^\pi(1-\cos^2u)\sin u\,du\cdot\int_0^{2\pi}\cos^2v\,dv+$$

$$+R^3\int_0^\pi\cos^2u\sin u\,du\cdot 2\pi=$$

$$=2\pi R^3\left[\tfrac{1}{3}\cos^3u-\cos u\right]_0^\pi+$$

$$+2\pi R^3\left[-\tfrac{1}{3}\cos^3u\right]_0^\pi=\underline{4\pi R^3.}$$

Anmärkning: $\int_0^{2\pi}\cos^3v\,dv=\int_0^{2\pi}\cos^2v\cos v\,dv=$

$$=\int_0^{2\pi}(1-\sin^2v)\cos v\,dv=\left(\int_0^{\pi/2}+\int_{\pi/2}^{3\pi/2}+\int_{3\pi/2}^{2\pi}\right)\cos^3v\,dv=$$

$$=\tfrac{2}{3}+\left(-\tfrac{1}{3}\right)+\tfrac{2}{3}=0.$$

b) $\mathbf{F}(x,y,z)=(x^2,2y,z)\Rightarrow \operatorname{div}\mathbf{F}=2x+3.$

Med $S: x^2+y^2+z^2=R^2$ och $V: x^2+y^2+z^2\le R^2$ fås

$$\oint\!\!\!\oint_S\mathbf{F}\cdot\hat{n}\,dS=\iiint_V(2x+3)\,dx\,dy\,dz=2\iiint_V x\,dx\,dy\,dz+$$

$$+3\iiint_V dx\,dy\,dz=0+3\cdot\tfrac{1}{3}\pi R^3=4\pi R^3.$$

Anm $\iiint_V x\,dx\,dy\,dz=\iint\limits_{y^2+z^2\le1}\left(\int\limits_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}}x\,dx\right)dy\,dz$

$$=\iint\limits_{y^2+z^2\le1}\left(\left[\tfrac{1}{2}x^2\right]_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}}\right)dy\,dz=0.$$

Uppgift 6.22 (Sid. 7)

Lösning

$$\mathbf{A}(x,y,z)=\frac{1}{\sqrt{x^2+y^2}}(x,1,0)$$

$0\le z\le1\Leftrightarrow 0\le-\sqrt{x^2+y^2}+1\le1\Leftrightarrow \underline{0\le\sqrt{x^2+y^2}\le1}$;

$S: \mathbf{r}(u,v)=(u\cos v,\ u\sin v,\ 1-u),\ 0\le u\le1,\ 0\le v\le2\pi.$

$$\mathbf{A}\cdot\hat{n}\,dS\longrightarrow\mathbf{A}(\mathbf{r}(u,v))\cdot(\mathbf{r}'_u\times\mathbf{r}'_v)\,du\,dv=$$

$$=\frac{1}{u}(u\cos v,1,0)\cdot u(\cos v,\sin v,1)\,du\,dv$$

$$=(u\cos^2v+\sin v)\,du\,dv\ \Rightarrow$$

$$\iint_S A\cdot\hat{n}\,dS=\int_0^1 du\int_0^{2\pi}(u\cos^2 v+\sin v)\,dv=$$
$$=\int_0^1 \pi u^2\,du=\frac{\pi}{3}.$$

Uppgift 6.23 (Sid. 7)

Lösning

$$\underline{F(x,y,z)=\frac{1}{r^3}(x,y,z)}$$

Jag bestämmer flödet medelst Gauss' sats.

S:s projektion i planet $z=1$ är skivan

$$\sigma:\ x^2+y^2\le 1,\ z=1$$

Jag stänger S med σ och sätter $\Sigma=S+\sigma$;

$$K:\ 1\le z\le\sqrt{5-4x^2-4y^2},\ x^2+y^2\le 1;\ \Sigma=\partial K$$

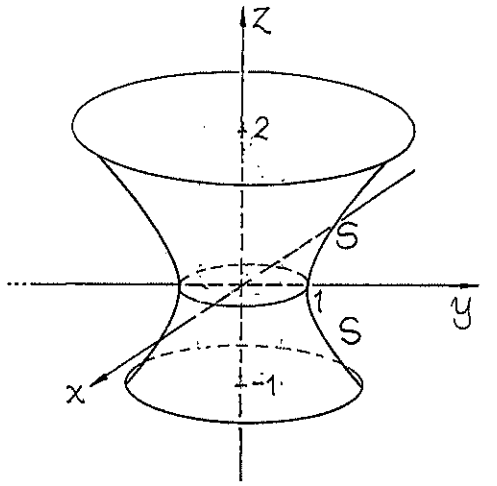
$\operatorname{div}F=\operatorname{div}(r^{-3}\mathbf{r})=-3r^{-4}\hat{r}\cdot\mathbf{r}+r^{-3}\cdot 3=-3r^{-3}+3r^{-3}=0.$

Se uppgift 4.7 d).

$$\oiint_\Sigma F\cdot\hat{n}\,dS=\iiint_K \operatorname{div}F\,dV=0\Leftrightarrow\iint_S A\cdot\hat{n}\,dS=$$
$$=-\iint_\sigma A\cdot(-\hat{z})\,dS=\iint_\sigma\frac{1}{(x^2+y^2+1)^{3/2}}\,dx\,dy=\ /\text{ty } z=1/=$$
$$=\int_0^1\frac{r}{(r^2+1)^{3/2}}\,dr\int_0^{2\pi}d\theta=2\pi\left[-\frac{1}{\sqrt{r^2+1}}\right]_0^1=\underline{\pi(2-\sqrt{2})}.$$

Uppgift 6.24 (Sid. 7)

Lösning



$$F(x,y,z)=\frac{(x,y,z)}{x^2+y^2}$$

Fältet är radiellt (cylindersymmetriskt).

så flödet genom de plana ytorna är 0;

En parametrisering av den krökta ytan är

$$S:\ r(u,v)=(\sqrt{u^2+1}\cos v,\ \sqrt{u^2+1}\sin v,\ u),\ \Delta:\begin{cases}-1\le u\le 2\\ 0\le v\le 2\pi\end{cases};$$

$$r_u'\times r_v'=\left(\frac{u\cos v}{\sqrt{u^2+1}},\ \frac{u\sin v}{\sqrt{u^2+1}},\ 1\right)\times(-\sqrt{u^2+1}\sin v,\ \sqrt{u^2+1}\cos v,\ 0)$$
$$=(-\sqrt{u^2+1}\cos v,\ -\sqrt{u^2+1}\sin v,\ u)\ \text{inåtriktad}.$$

$\mathcal{F}(r(u,v)) = \left(\frac{\cos v}{\sqrt{u^2+1}}, \frac{\sin v}{\sqrt{u^2+1}}, 0\right) \Rightarrow A\cdot\hat{n}\,dS \longrightarrow du\,dv \Rightarrow$

$\Rightarrow \iint_S A\cdot\hat{n}\,dS = \iint_\Delta du\,dv = 2\pi\cdot 3 = \underline{6\pi}.$

Uppgift 6.25 (Sid. 7)

Lösning

$u = \sqrt{x^2+y^2+z^2} \Rightarrow \operatorname{grad} f(x,y,z) = -2u^{-3}\hat{u}$, där

$\hat{u} = \frac{(x,y,z)}{u}$ är en utåtriktad

$$\oiint_{u=r} \operatorname{grad}\frac{1}{u^2}\cdot\hat{n}\,dS = -2\oiint_{u=r} r^{-3}\hat{u}\cdot\hat{u}\,dS = -\frac{2}{r^3}\oiint_{u=r} dS =$$

$$= -\frac{2}{r^3}\cdot 4\pi r^2 = \underline{-\frac{8\pi}{r}}.$$

Uppgift 6.26 (Sid. 7)

Lösning

$S: r(u,v) = (2\cos u, \sin u, v+1),\quad 0\le u\le 2\pi,\ 0\le v\le 1.$

$r'_u \times r'_v = (-2\sin u, \cos u, 0)\times(0,0,1) = (\cos u, 2\sin u, 0)$

$$\left.\begin{aligned} A(x,y,z) &= \left(-\frac{6x}{x^2+y^2}, -\frac{6y}{x^2+y^2}, z+1\right);\\ A(r(u,v)) &= \left(-\frac{12\cos u}{4\cos^2 v+\sin^2 v}, -\frac{6\sin u}{4\cos^2 u+\sin^2 u}, v+1\right)\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow A\cdot\hat{n}\,dS \longrightarrow A(r(u,v))\cdot(r'_u\times r'_v)\,du\,dv =$$

$$= -\frac{12}{4\cos^2 u+\sin^2 u}\,du\,dv =$$

$$= -\frac{12}{3\cos^2 u+1}\,du\,dv =$$

$$= -\frac{12}{\frac{3}{2}(1+\cos 2v)+1}\,du\,dv =$$

$$= -\frac{24}{5+3\cos 2v}\,du\,dv \Rightarrow$$

$$\iint_S A\cdot\hat{n}\,dS = -24\int_0^{2\pi}\frac{1}{5+3\cos 2u}\,du\int_0^1 dv =$$

$$= -24\int_0^{2\pi}\frac{du}{5+3\cos 2u} = /\text{perioden } \pi/ =$$

$$= -48\int_0^{\pi}\frac{du}{5+3\cos 2u} = /2u = w/ =$$

$$= -24\int_0^{2\pi}\frac{dw}{5+3\cos w} = -48\int_0^{\pi}\frac{dw}{5+3\cos w} =$$

$$= -48\cdot\frac{1}{2}\left[\arctan\frac{2\tan(x/2)}{4}\right]_0^{\pi} =$$

$$= -24\cdot\frac{\pi}{2} = \underline{-12\pi}.$$

Anm. $f(u) = \frac{1}{5+3\cos 2u} \Rightarrow f(u+\pi) = f(u) \Rightarrow \int_0^{2\pi} f =$

$= \left(\int_0^{\pi} + \int_{\pi}^{2\pi}\right) f(u)\,du = 2\int_0^{\pi} f(u)\,du$. P.s.s. fås för

$g(w) = \frac{1}{5+3\cos w} = g(-w)$ att $\int_0^{2\pi} g = \left(\int_0^{\pi} + \int_{-\pi}^{0}\right) g = 2\int_0^{\pi} g.$

Uppgift 6.27 (Sid. 7)

Lösning

$$A(x,y,z)=(x^3,y^3,z(z-1)e^{x^2+y^2}).$$

$S:\ r(u,v)=(\cos u,\sin u,v);\ \Delta: 0\le u\le 2\pi,\ 0\le v\le 1$

$$r_u'\times r_v'=(-\sin u,\cos u,0)\times(0,0,1)=(\cos u,\sin u,0);$$

$$A\cdot\hat{n}\,dS \longrightarrow A(r(u,v))\cdot(r_u'\times r_v')\,du\,dv=$$

$$=(\cos^3 u,\sin^3 u,ev(v-1))\cdot(\cos u,\sin u,0)\,du\,dv$$

$$=(\cos^4 u+\sin^4 u)\,du\,dv=$$

$$=((\cos^2 u)^2+(\sin^2 u)^2)\,du\,dv=$$

$$=\left(\left(\frac{1+\cos 2u}{2}\right)^2+\left(\frac{1-\cos 2u}{2}\right)^2\right)du\,dv$$

$$=\frac{1}{4}(2+2\cos^2 2u)\,du\,dv=$$

$$=\frac{1}{4}(3+\cos 4u)\,du\,dv \Rightarrow$$

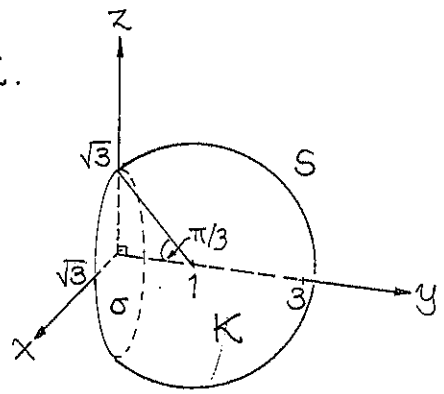
$$\iint_S A\cdot\hat{n}\,dS=\frac{1}{4}\int_0^{2\pi}(3+\cos 4u)\,du\int_0^1 dv=\frac{1}{4}\cdot 6\pi=\frac{3\pi}{2}.$$

Uppgift 6.28 (Sid. 7)

Lösning

$$\operatorname{div}\mathbf{F}=\partial_x(2xy+x^2)+\partial_y(2+yz)+\partial_z(2z^4)=2x+2y+z+8z^3.$$

$$\Sigma=S+\sigma=\partial K.$$



$$x^2+(y-1)^2+z^2=4 \Rightarrow /y=0/ \Rightarrow x^2+z^2=3$$

Jag stänger S med σ för att tillämpa Gauss' sats på det stympade klotet (se ovan).

$$\begin{cases} x=r\sin\theta\sin\varphi \\ y=1+r\cos\theta \\ z=r\sin\theta\cos\varphi \end{cases};\quad \Delta: 0\le r\le 2,\ 0\le\theta\le\frac{2\pi}{3},\ 0\le\varphi\le 2\pi$$

$$dV=dx\,dy\,dz \longrightarrow r^2\sin\theta\,dr\,d\theta\,d\varphi.$$

$$\oiint_\Sigma A\cdot\hat{n}\,dS=\iiint_K(2x+2y+z+8z^3)\,dV=2\iiint_K y\,dV=$$

$$=2\int_0^3\pi y(x^2+z^2)\,dy=2\pi\int_0^3 y(4-(y-1)^2)\,dy=$$

$$=2\int_0^3\pi(3y+2y^2-y^3)\,dy=2\left[\frac{3}{2}y^2+\frac{2}{3}y^3-\frac{1}{4}y^4\right]_0^3\pi=$$

$$=2\left(\frac{27}{2}+18-\frac{81}{4}\right)\pi=\left(63-\frac{81}{2}\right)\pi=\frac{45}{2}\pi \Leftrightarrow$$

$$\iint_S A\cdot\hat{n}\,dS+\iint_\sigma A\cdot(-\hat{y})\,dS=\frac{45}{2}\pi \Leftrightarrow \iint_S A\cdot\hat{n}\,dS=\frac{45}{2}\pi+$$

$$+\iint_\sigma A\cdot\hat{y}\,dS = \frac{45\pi}{2} + 2\iint_\sigma dS = \frac{45\pi}{2} + 2\cdot 3\pi = \frac{57\pi}{2}.$$

Anm. På "locket" σ är $y=0$, s.a. $A(\sigma)=(0,2,0)$

Integrationselementet kan väljas godtyckligt; jag har ovan använt "skivformeln":

$dV = \pi(x^2+y^2)\,dy = \pi(4-(y-1)^2)\,dy$.

K speglar i yz- och xy-planet så vi får

$$\int_K x\,dV = \int_K z\,dV = \int_K z^3\,dV = 0 \quad (*)$$

Integralerna i (*) är trippelintegraler.

7. Kurvintegraler, Stokes sats

Uppgift 7.1 (Sid. 7)

Lösning

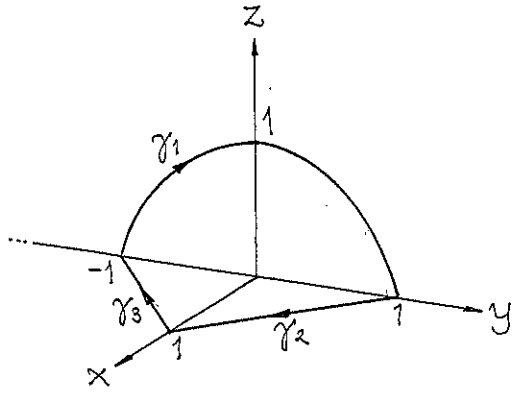
γ: $r(t)=(t,t^2,t^3)$, $0\le t\le 1$; $F(x,y,z)=(\sqrt{y},2x,3y)$

$$\int_\gamma F\cdot dr = \int_0^1 F(r(t))\cdot r'(t)\,dt = \int_0^1 (t,2t,3t^2)\cdot(1,2t,3t^2)\,dt =$$
$$= \int_0^1 (t+4t^2+9t^4)\,dt = \left[\frac{t^2}{2}+\frac{4t^3}{3}+\frac{9t^5}{5}\right]_0^1 = \frac{109}{30}.$$

Uppgift 7.2 (Sid. 8)

Lösning

$F(x,y,z)=(xz, xy^2+2z, xy+z)$



$\gamma_1+\gamma_2+\gamma_3=\gamma$

(1) γ_1: $r(t)=(0,-\cos t,\sin t)$, $0\le t\le\pi$.

$F(r(t))=(0,2\sin t,\sin t)$; $\frac{dr}{dt}=(0,\sin t,\cos t)$;

$$\int_{\gamma_1} F\cdot dr = \int_0^\pi (0,2\sin t,\sin t)\cdot(0,\sin t,\cos t)\,dt =$$
$$= \int_0^\pi (2\sin^2 t+\sin t\cos t)\,dt =$$
$$= \int_0^\pi (1-\cos 2t+\tfrac{1}{2}\sin 2t)\,dt = \pi$$

(2) γ_2: $r(t)=(t,1-t,0)$, $0\le t\le 1$.

$F(r(t))=(0,t(1-t)^2,t(1-t))$; $\frac{dr}{dt}=(1,-1,0)$;

$$\int_{\gamma_2} F\cdot dr = \int_0^1 t(1-t)^2\,dt = \int_0^1 (t-2t^2+t^3)\,dt = -\frac{1}{12}$$

(3) $\gamma_3: r(t) = (1-t, -t, 0),\ 0 \le t \le 1.$

$F(r(t)) = (0, (1-t)t^2, t(t-1))$; $\frac{dr}{dt} = (-1,-1,0)$;

$\int_{\gamma_3} F \cdot dr = \int_0^1 t^2(t-1)\,dt = \int_0^1 (t^3 - t^2)\,dt = -\frac{1}{12}$.

$\oint_\gamma F \cdot dr = \int_{\gamma_1} F\cdot dr + \int_{\gamma_2} F\cdot dr + \int_{\gamma_3} F\cdot dr = \pi - \frac{1}{12} - \frac{1}{12} = \pi - \frac{1}{6}$.

Uppgift 7.3 (Sid. 8)

Lösning

Jag kommer att tillämpa Stokes' sats.

$F(x,y,z) = (3x^2z,\ y^3 - x^3,\ 2y+x)$;

$$\operatorname{rot} F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 3x^2z & y^3-x^3 & 2y+x \end{vmatrix} = (2,\ 3x^2-1,\ -3x^2).$$

$S: r(u,v) = (u, v, -u-v),\ \Delta: u^2+v^2 \le 1.$

$r_u' \times r_v' = (1,1,1) \Rightarrow \oint_\gamma F\cdot dr = \iint_\Delta (\operatorname{rot} F)\cdot(1,1,1)\,du\,dv$

$= \iint_\Delta (2,\ 3u^2-1,\ -3u^2)\cdot(1,1,1)\,dv = \iint_\Delta du\,dv = \mathcal{A}(D) = \pi$.

Som linjeintegral blir den svårare; krångliga fältuttryck medför... Stokes sats.

Uppgift 7.4 (Sid. 8)

Lösning

(1) Som linjeintegral

$0 \le z \le 1 \Leftrightarrow 0 \le 1 - (x^2+y^2)^{3/2} \le 1 \Leftrightarrow \Delta: x^2+y^2 \le 1.$

$\gamma: r(t) = (\cos t, \sin t, 0),\ 0 \le t \le 2\pi.$

$$\int_\gamma F\cdot dr = \int_0^{2\pi} F(r(t))\cdot r'(t)\,dt =$$

$$= \int_0^{2\pi} (0, \cos t, \sin t)\cdot(-\sin t, \cos t, 0)\,dt =$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi.$$

(2) Som ytintegral

$\operatorname{rot} F = (1,1,1)$;

$S: r(u,v) = (u\cos v, u\sin v, 1-u^3)$; $\Delta: \begin{cases} 0 \le u \le 1 \\ 0 \le v \le 2\pi \end{cases}$;

$(\operatorname{rot} F)\cdot \hat{n}\,dS = (1,1,1)\cdot(3u^3\cos v, 3u^3\sin v, u)\,du\,dv$

$$= (3u^3(\cos v + \sin v) + u)\,du\,dv.$$

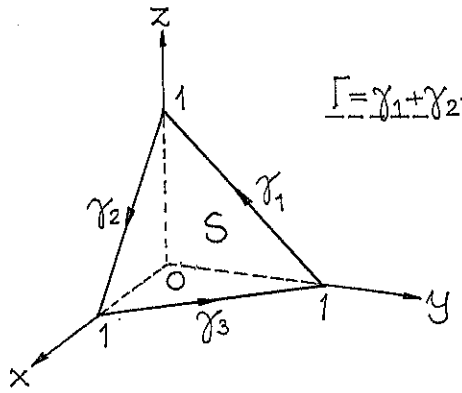
$$\iint_S (\operatorname{rot} F)\cdot\hat{n}\,dS = \int_0^1 du \int_0^{2\pi} (3u^2(\cos v + \sin v) + u)\,dv$$

$$= \int_0^1 \left(\int_0^{2\pi} u\,dv\right) du = \int_0^1 u\,du \int_0^{2\pi} dv = \pi.$$

Därmed är Stokes sats verifierad.

Uppgift 7.5 (Sid. 8)

Lösning: $\mathbf{F}(x,y,z) = (x^2,\ x^2+y^2,\ x^2+y^2+z^2)$



$\Gamma = \gamma_1+\gamma_2+\gamma_3 = \partial S$

Jag kommer att använda Stokes sats:

$S: \mathbf{r}(u,v) = (u,v,1-u-v)$, $\Delta: 0\le u\le 1,\ 0\le v\le 1-u$.

$\mathrm{rot}\,\mathbf{F} = (2y,-2x,2x)$; $\mathbf{r}'_u \times \mathbf{r}'_v = (1,1,1)$.

$(\mathrm{rot}\,\mathbf{F})\cdot\hat{n}\,dS = (2v,-2u,+2u)\cdot(1,1,1)\,du\,dv = 2v\,du\,dv$

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \iint_\Delta 2v\,du\,dv = \int_0^1 du\int_0^{1-u} 2v\,dv =$$

$$= \int_0^1 (1-u)^2\,du = \left[-\tfrac{1}{3}(1-u)^2\right]_0^1 = \frac{1}{3}.$$

Anm Denna integral kan bestämmas som linjeintegral. Låt mig göra det och på så sätt verifiera Stokes' sats.

(1) $\gamma_1: \mathbf{r}(t) = (0,1-t,t),\ 0\le t\le 1$

$\mathbf{F}\cdot d\mathbf{r} = (0,\ (1-t)^2,\ (1-t)^2+t^2)\cdot(0,-1,1)\,dt = t^2\,dt$;

$\int_{\gamma_1}\mathbf{F}\cdot d\mathbf{r} = \int_0^1 t^2\,dt = \frac{1}{3}$.

(2) $\gamma_2: \mathbf{r}(t) = (t,0,1-t),\ 0\le t\le 1$.

$\mathbf{F}\cdot d\mathbf{r} = (t^2,\ t^2,\ t^2+(1-t)^2)\cdot(1,0,-1)\,dt = -(1-t)^2\,dt$;

$\int_{\gamma_2}\mathbf{F}\cdot d\mathbf{r} = -\int_0^1 (1-t)^2\,dt = \left[\frac{1}{3}(1-t)^3\right]_0^1 = -\frac{1}{3}$;

(3) $\gamma_3: \mathbf{r}(t) = (1-t,t,0),\ 0\le t\le 1$.

$\mathbf{F}\cdot d\mathbf{r} = ((1-t)^2,\ (1-t)^2+t^2,\ (1-t)^2+t^2)\cdot(-1,1,0)\,dt =$

$= t^2\,dt$;

$\int_{\gamma_3}\mathbf{F}\cdot d\mathbf{r} = \int_0^1 t^2\,dt = \frac{1}{3}$.

(4) $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_2}\right)\mathbf{F}\cdot d\mathbf{r} = \frac{1}{3}-\frac{1}{3}+\frac{1}{3} = \frac{1}{3}$.

Resultat: $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \iint_S (\mathrm{rot}\,\mathbf{F})\cdot\hat{n}\,dS = \frac{1}{3}$.

Uppgift 7.6 (Sid. 8)

Lösning

Man kan bestämma integralen som linje-

integral; jag räknar med Stokes sats, dvs använder ytintegration.

$\mathbf{F}(x,y,z)=(xz, yz, x^3y^4) \Rightarrow \operatorname{rot}\mathbf{F}=(4x^3y^3-y, x-3x^2y^4, 0)$.

S: $\mathbf{r}(u,v)=(u,v,3u+4v)$, Δ: $0\le u,v\le 1$.

$\mathbf{r}'_u\times\mathbf{r}'_v=(1,0,3)\times(0,1,4)=(3,4,-1)$;

$(\mathbf{r}'_u\times\mathbf{r}'_v)\cdot(0,0,10)=-10 \Rightarrow$ jag använder $-\mathbf{r}'_u\times\mathbf{r}'_v$.

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} \overset{!}{=} -\iint_\Delta \operatorname{rot}\mathbf{F}\cdot \mathbf{r}'_u\times\mathbf{r}'_v\,du\,dv =$$

$$= -\iint_\Delta (12u^3v^3-12u^2v+4u-3v)\,du\,dv =$$

$$= \iint_\Delta 12u^2v^5\,dv = 12\int_{-1}^{1}u^2\,du\int_{-1}^{1}v^4\,dv =$$

$$= 12\cdot\frac{2}{3}\cdot\frac{2}{5}=\frac{16}{5}.$$

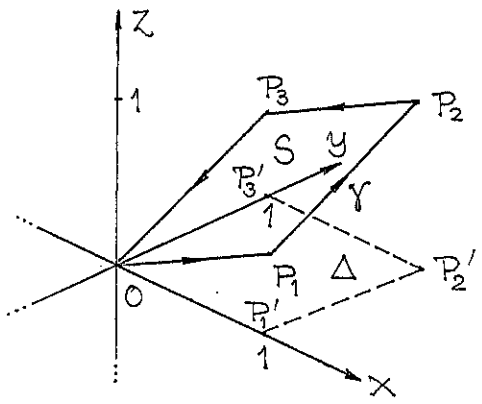
Anm. $\int_{-1}^{1}u^3du=\int_{-1}^{1}v^3dv=\int_{-1}^{1}u\,du=\int_{-1}^{1}v\,dv=0$.

Uppgift 7.7 (Sid. 8)

Lösning

$P_1=(1,0,1/2)$, $P_2=(1,1,1)$, $P_3=(0,1,1/2)$;

$\mathbf{F}(x,y,z)=(e^{x^2}+y^2, e^{y^2}+2z^2, e^{z^2}+3x^2)$;



Jag bestämmer ekvationen för planet gm punkterna P_1, P_2 och P_3. Observera att $\overrightarrow{OP_1}+\overrightarrow{OP_3}=\overrightarrow{OP_2}$, dvs origo och P_1, P_2, P_3 ligger är komplana.

$n=\overrightarrow{OP_1}\times\overrightarrow{OP_3}=(1,0,\frac{1}{2})\times(0,1,\frac{1}{2})=(-\frac{1}{2},-\frac{1}{2},1)$;

Planets ekvation är alltså

$$\pi: -\tfrac{1}{2}x-\tfrac{1}{2}y+z=0.$$

Jag behöver vidare S:s projektion i xy-planet.

$P_1'=(1,0,0)$, $P_2'=(1,1,0)$, $P_3'=(0,1,0)$;

$\operatorname{rot}\mathbf{F}=(-4z,-6x,-2y)$;

S: $\mathbf{r}(u,v)=(u,v,\frac{1}{2}u+\frac{1}{2}v)$, Δ: $0\le u,v\le 1$ (se ovan).

$r'_u \times r'_v = (1,0,\frac{1}{2}) \times (0,1,\frac{1}{2}) = (-\frac{1}{2},-\frac{1}{2},1)$;

$\int_\gamma F \cdot dr = \iint_S (-4z,-6x,-2y)\cdot(-\frac{1}{2},-\frac{1}{2},1)\,dS =$

$= \iint_S (2z+3x-2y)\,ds =$

$= \iint_\Delta (u+v+3u-2v)\,du dv =$

$= \iint_\Delta (4u-v)\,du dv = 2 - \frac{1}{2} = \frac{3}{2}.$

Uppgift 7.8 (Sid. 8)

Lösning

Jag kommer att använda Stokes' sats.

Kurvans projektion i xy-planet bestäms:

z-koordinaterna lika $\Rightarrow z = \frac{1-y}{2} = 2 - \sqrt{x^2+y^2} \Rightarrow$

$1-y = 4 - 2\sqrt{x^2+y^2} \Leftrightarrow 2\sqrt{x^2+y^2} = y+3 \Rightarrow 4(x^2+y^2) =$

$= y^2+6y+9 \Leftrightarrow 4x^2+3y^2-6y = 9 \Leftrightarrow 4x^2+3(y-1)^2 = 12$

$\Leftrightarrow \frac{x^2}{3} + \frac{(y-1)^2}{4} = 1.$

$A(x,y,z) = (-y,x,z) \Rightarrow \operatorname{rot} A = (0,0,2)$

$S: r(u,v) = (u, v, \frac{1}{2} - \frac{1}{2}v)$; $\Delta: \frac{u^2}{3} + \frac{(v-1)^2}{4} \le 1.$

Observera att alla C^1-ytor med γ till rand ger samma ytintegral.

$r'_u \times r'_v = (1,0,0) \times (0,1,-\frac{1}{2}) = (0,\frac{1}{2},1)$

$\oint_\gamma F \cdot dr = \iint_\Delta (0,0,2)\cdot(0,\frac{1}{2},1)\,du dv = 2\iint_\Delta du dv =$

$= 2 \cdot \pi \cdot \sqrt{3} \cdot 2 = \underline{4\sqrt{3}\pi}.$

Anm. $D: \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} \le 1 \Rightarrow A(D) = \pi \cdot ab.$

Uppgift 7.9 (Sid. 8)

Lösning

Jag kommer att använda Stokes sats.

$F(x,y,z) = (x+y+z,\ xyz,\ z^2) \Rightarrow \underline{\operatorname{rot} F = (-xy,\ 1,\ yz-1)}$

$\underline{S: r(u,v) = (u, v, 5-\sqrt{5}u)}$, $\underline{\Delta: u^2+v^2 \le 5.}$

$\int_\gamma F \cdot dr = \iint_\Delta \operatorname{rot} F \cdot (r'_u \times r'_v)\,du dv =$

$= \iint_\Delta (-uv,\ 1,\ v(5-\sqrt{5}u)-1)\cdot(\sqrt{5},0,1)\,du dv$

$= \iint_\Delta (-\sqrt{5}uv + 5v - \sqrt{5}uv - 1)\,du dv =$

$= \iint_\Delta (5-2\sqrt{5}u)v\,du dv - \iint_\Delta du dv =$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} du\,(5-2\sqrt{5}u) \int_{-\sqrt{5-u^2}}^{\sqrt{5-u^2}} v\,dv - \mathcal{A}(\Delta) = 0 - 5\pi = \underline{-5\pi}.$$

Problem 7.10 (Sid. 8)

Lösning

Jag kommer att använda Stokes sats så jag behöver ytstyckets projektion i xy-planet; z-koordinaterna lika gör mig

$x^2+y^2 = 1-2x-2y \Leftrightarrow \underline{\Delta: (x+1)^2+(y+1)^2 = 3 = (\sqrt{3})^2}.$

$\mathcal{F}(x,y,z) = ((z-1)^3, x^3, y^3) \Rightarrow \underline{\operatorname{rot}\mathcal{F} = (3y^2, 3(z-1)^2, 3x^2)}.$

$(0,0,20)\cdot(2,2,1) = 20 > 0 \Rightarrow \underline{\Gamma \text{ pos. orienterad}}.$

$\underline{S: r(u,v) = (u, v, 1-2u-2v)}, \ \underline{\Delta: u^2+v^2 \le 3}.$

$$\int_{\Gamma} \mathcal{F}\cdot dr = \iint_{\Delta} (3v^2, 12(u+v)^2, 3u^2)\cdot(2,2,1)\,du\,dv =$$

$$= \iint_{\Delta} (6v^2 + 24(u+v)^2 + 3u^2)\,du\,dv =$$

$$= \iint_{\Delta} (27u^2 + 30v^2 + 48uv)\,du\,dv =$$

$$= \left[\begin{array}{l|l} u = -1+r\cos\theta & 0 \le r \le \sqrt{3} \\ v = -1+r\sin\theta & 0 \le \theta \le 2\pi \end{array} \middle| \; du\,dv \to r\,dr\,d\theta \right] =$$

$$= \int_0^{\sqrt{3}} dr\, r \int_0^{2\pi} (105 + 27r^2\cos^2\theta + 30r^2\sin^2\theta -$$

$$-108r(\sin\theta+\cos\theta) + 24r^2\sin 2\theta)\,d\theta =$$

$$= \int_0^{\sqrt{3}} (210\pi + 57r^2\pi) r\,dr = 3\pi \int_0^{\sqrt{3}} (70r + 19r^3)\,dr =$$

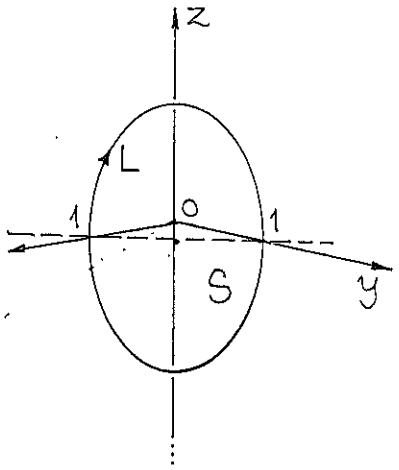
$$= 3\pi \left[35r^2 + \frac{19r^4}{4}\right]_0^{\sqrt{3}} = 3\pi\left(105 + \frac{171}{4}\right) = \underline{\frac{1773}{4}\pi}.$$

Anm. $\int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} \sin^2 x\,dx = \pi.$

$\int_0^{2\pi} \cos x\,dx = \int_0^{2\pi} \sin x\,dx = \int_0^{2\pi} \sin 2x\,dx = 0.$

Uppgift 7.11 (Sid. 8)

Lösning



$S: \begin{cases} x+y=1 \\ x^2+y^2+z^2 \le 1 \end{cases}$

Jag kommer att tillämpa Stokes sats:

$\mathcal{F}(x,y,z) = (yz+y-z, xz+5x, xy+2y) \Rightarrow$

$\text{rot}\,\mathbf{F} = (2,-1,4) \Rightarrow \int_L \mathbf{F}\cdot d\mathbf{r} = \iint_S (2,-1,4)\cdot(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)\,dS$

$= -\frac{1}{\sqrt{2}}\cdot \mathcal{A}(S) = -\frac{\sqrt{2}}{2}\cdot\pi\left(\frac{\sqrt{2}}{2}\right)^2 = -\frac{\pi\sqrt{2}}{4}.$

Uppgift 7.12 (Sid. 9)

Lösning

Jag kommer att tillämpa Stokes sats, så jag behöver S:s projektion i xy-planet.

$z = y-1 \Rightarrow 2x^2+2y^2 = (y-1)^2 \Leftrightarrow 2x^2+2y^2 = y^2-2y+1$

$\Leftrightarrow 2x^2+y^2+2y+1 = 2 \Leftrightarrow 2x^2+(y+1)^2 = 2$

$\Delta: x^2+\frac{(y+1)^2}{(\sqrt{2})^2} \le 1.$

$S: \mathbf{r}(u,v) = (u, v, v-1),\ (u,v)\in\Delta$ (se ovan).

$\mathbf{F} = (y,0,1) \Rightarrow \text{rot}\,\mathbf{F} = (0,0,-1)$

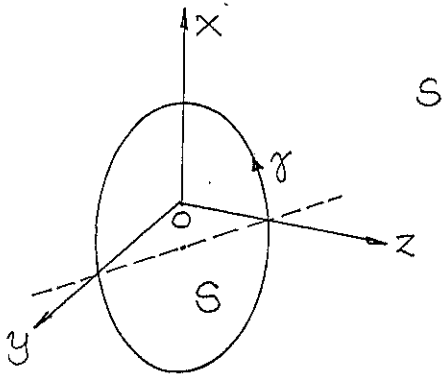
$\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = \iint_S (0,0,-1)\cdot(0,-1,1)\,dS = -\iint_\Delta du\,dv = -\pi\sqrt{2}.$

Anm. $(0,0,20)\cdot(0,1,-1) = -20 < 0$, så jag väljer $\hat{n} = -(0,1,-1)$ i ytintegralen i Stokes sats.

Alla ytor som är $\mathcal{C}^1$ duger; jag valde planet.

Uppgift 7.13 (Sid. 9)

Lösning



$S: \begin{cases} y+z=1 \\ x^2+y^2+z^2 \le 1 \end{cases}$

$\mathbf{F}(x,y,z) = (1-2y, 2x, 0) \Rightarrow \text{rot}\,\mathbf{F} = (0,0,4) \Rightarrow$

$\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = \iint_S (0,0,4)\cdot(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\,dS = \frac{4}{\sqrt{2}}\cdot\mathcal{A}(S) = \sqrt{2}\cdot\pi.$

Uppgift 7.14 (Sid. 9)

Lösning

(1) $x^2+y^2 = x \Leftrightarrow (x-\frac{1}{2})^2+y^2 = \frac{1}{4}$; $z = x^2+y^2$.

$\Gamma: \mathbf{r}(t) = \frac{1}{2}(1+\cos t, \sin t, 1-\cos t),\ 0 \le t \le 2\pi.$

(2) $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \frac{1}{4}(\sin t, 1, 1+\cos t)\cdot(-\sin t, \cos t, \sin t)\,dt$

$= \frac{1}{4}\int_0^{2\pi} (-\sin^2 t + \cos t + \sin t + \sin t\cos t)\,dt =$

$= \frac{1}{4}\cdot(-\pi) = -\frac{\pi}{4}.$ (Se Anm. till Ö. 7.10).

(3) $S: r(x,y) = (x, y, x^2+y^2)$, $D: (x-\frac{1}{2})^2+y^2 = \frac{1}{4}$.

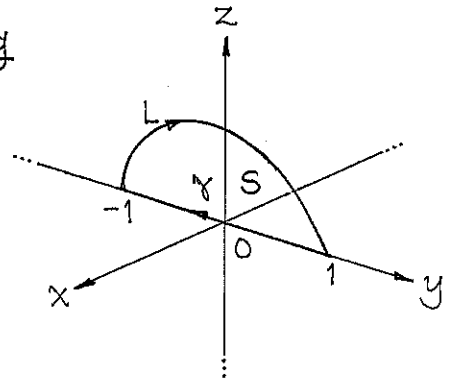
$$\left.\begin{array}{l} F(x,y,z) = (y,1,x) \Rightarrow \operatorname{rot} F = (0,1,-1) \\ r'_u \times r'_v = (1,0,2x)\times(0,1,2y) = (-2x,-2y,1) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \oint_\Gamma F\cdot dr = \iint_D (0,1,-1)\cdot(-2x,-2y,1)\,dx\,dy =$$

$$= \iint_D (-1-2y)\,dx\,dy \overset{!}{=} -\iint_D dx\,dy = -\frac{\pi}{4}.$$

Uppgift 7.15 (Sid. 9)

Lösning



$\Gamma = L+$

$$z = x \Leftrightarrow x-z = 0 \Rightarrow \hat{n} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \Rightarrow \hat{v} = -\hat{n} \Rightarrow$$

$$\oint_\Gamma F\cdot dr = \oint_\Gamma (z^2, y^2, 2xz)\cdot dr \overset{!}{=} \iint_S 0\cdot\hat{v}\,dS = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_L F\cdot dr + \int_\gamma F\cdot dr = 0 \Leftrightarrow \int_\Gamma F\cdot dr = \int_{-\gamma} F\cdot dr =$$

$$= \int_{-1}^{1} y^2\,dy = 2\int_0^1 y^2\,dy = 2\cdot\frac{1}{3} = \frac{2}{3}.$$

Anm. $\overset{!}{=}$ tas $\hat{v} = \frac{1}{\sqrt{2}}(-1,0,1)$ för pos. riktning.

Uppgift 7.16 (Sid. 9)

Lösning

$$\operatorname{rot} A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z, 2z-2x, 2x-2y)$$

$\hat{n} = \frac{1}{R}(x,y,z)$ är en enhetsnormal till sfären;

Stokes sats ger

$$\oint_\gamma A\cdot dr = \iint_S \operatorname{rot} A\cdot\hat{n}\,dS =$$

$$= \iint_S 2(y-z, z-x, x-y)\cdot\frac{1}{R}(x,y,z)\,dS =$$

$$= \frac{2}{R}\iint_S 0\cdot dS = 0, \quad \text{V.S.V.}$$

γ är är en enkel sluten kurva och S är en "skärva" av det sfäriska skalet som omsluts av γ.

Uppgift 7.17 (Sid. 9)

Lösning

a) γ omsluter inte z-axeln ($x=y=0$) så jag

kan tillämpa Stokes sats.

$$\oint_\gamma A\cdot dr=\iint_S(\operatorname{rot}A)\cdot\hat{n}\,dS=\iint_S 0\cdot\hat{n}\,dS=0.$$

b) γ omsluter z-axeln; Stokes sats kan inte användas denna gång; linjeintegral!

$$\underline{\gamma:\ r(t)=(\cos t,\sin t,-\cos t),\ 0\le t\le 2\pi.}$$

$$A(x,y,z)=\left(-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2},0\right);$$

$$\left.\begin{aligned}A(r(t))&=(-\sin t,\cos t,0)\\ dr=r'(t)\,dt&=(-\sin t,\cos t,\sin t)\,dt\end{aligned}\right\}\Rightarrow\oint_\gamma A\cdot dr=$$

$$=\int_0^{2\pi}(\sin^2t+\cos^2t)\,dt=\int_0^{2\pi}dt=\underline{2\pi.}$$

Uppgift 7.18 (Sid. 9)

Lösning

Det går alldeles utmärkt att räkna med Stokes sats. Jag bestämmer kurvans projektion i xy-planet; z-koordinaterna lika:

$$4x^2+4y+x^2+y^2=5(x^2+y^2)=40\Leftrightarrow\underline{x^2+y^2=8.}$$

$$F(x,y,z)=\left(\frac{y}{x^2+y^2}-z,\,x-\frac{x}{x^2+y^2},\,y+x^2\right)\Rightarrow$$

$$\operatorname{rot}F=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\ \partial_x&\partial_y&\partial_z\\ \frac{y}{x^2+y^2}-z&x-\frac{x}{x^2+y^2}&y+x^2\end{vmatrix}=(1,-1-2x,1)$$

$$4x^2+4y^2+z^2=40\Leftrightarrow\underline{z=2\sqrt{10-x^2-y^2}}$$

$$S:\ r(u,v)=(u\cos v,u\sin v,2\sqrt{10-u^2});\ \Delta:\begin{cases}0\le u\le\sqrt{8}\\ 0\le v\le 2\pi\end{cases};$$

$$r_u'\times r_v=\left(-\frac{2u^2\cos v}{\sqrt{10-u^2}},-\frac{2u^2\sin v}{\sqrt{10-u^2}},u\right),\ (u,v)\in\Delta.$$

$$\oint_\Gamma F\cdot dr=\iint_S(1,-1-2x,1)\cdot\hat{n}\,dS=\iint_S(1,-1,1)\cdot\hat{n}\,dS=$$

$$=\iint_\Delta\left(\frac{2u^2}{\sqrt{10-u^2}}(\sin v-\cos v)+u\right)du\,dv=$$

$$=\int_0^{\sqrt{8}}du\,\frac{2u^2}{\sqrt{10-u^2}}\int_0^{2\pi}(-\cos v+\sin v)\,dv+\iint_\Delta u\,du\,dv=$$

$$=\int_0^{\sqrt{8}}u\,du\int_0^{2\pi}dv=\frac{1}{2}\cdot 8\cdot 2\pi=\underline{8\pi.}$$

Uppgift 7.19 (Sid. 9)

Lösning

$$\underline{F(x,y,z)=\left(\frac{y}{x^2+y^2},-\frac{x}{x^2+y^2},x\right).}$$

γ omsluter z-axeln ($x=y=0$) så Stokes sats kan inte användas i detta fall.

$\gamma: r(t) = (\cos t, \sin t, \cos t),\ 0 \le t \le 2\pi.$

$F(r(t)) \cdot r'(t)\,dt = (\sin t, -\cos t, \cos t) \cdot (-\sin t, \cos t, -\sin t)\,dt$

$= (-\sin^2 t - \cos^2 t - \sin t \cos t)\,dt =$

$= -(1 + \frac{1}{2}\sin 2t)\,dt\ ;$

$\oint_\gamma F \cdot dr = \int_0^{2\pi} (-1 - \frac{1}{2}\sin 2t)\,dt = -2\pi.$

Uppgift 7.20 (Sid. 9)

Lösning

$x^2 + y^2 = x \Leftrightarrow (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}\ ;\ A = (y, 1, x).$

$\gamma: r(t) = (\frac{1}{2} + \frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{1}{2} - \frac{1}{2}\cos t),\ 0 \le t \le 2\pi.$

$\left.\begin{array}{l} A(r(t)) = (\frac{1}{2}\sin t, 1, \frac{1}{2} + \frac{1}{2}\cos t) \\ r'(t) = \frac{1}{2}(-\sin t, \cos t, \sin t) \end{array}\right\} \Rightarrow A(r(t)) \cdot r'(t)\,dt =$

$= \frac{1}{4}(\sin t, 2, 1 + \cos t) \cdot (-\sin t, \cos t, \sin t)\,dt =$

$= \frac{1}{4}(-\sin^2 t + 2\cos t + \sin t + \sin t \cos t)\,dt \Rightarrow$

$\oint_\gamma A \cdot dr = \frac{1}{4}\int_0^{2\pi} (-\sin^2 t + 2\cos t + \sin t + \sin t \cos t)\,dt =$

$= \frac{1}{4}\int_0^{2\pi} (-\sin^2 t)\,dt = \frac{1}{4} \cdot (-\pi) = -\frac{\pi}{4}.$

8. Potentialfält

Uppgift 8.1 (Sid. 9)

Lösning

a) $F(x,y) = (x\ln(1 + x^2 + y^2),\ y\ln(1 + x^2 + y^2)).$

$\begin{cases} P(x,y) = x\ln(1 + x^2 + y^2) \\ Q(x,y) = y\ln(1 + x^2 + y^2) \end{cases} \Rightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{2xy}{1 + x^2 + y^2} \\ \frac{\partial Q}{\partial x} = \frac{2xy}{1 + x^2 + y^2} \end{cases} \Rightarrow$

$\Rightarrow$ F är ett potentialfält. (Se s. 351 i boken).

Anm. I fortsättningen hänvisar jag till teoriboken: Persson & Böiers, Analys i flera variabler, Studentlitteratur, 2005.

b) $F(x,y) = (e^{x-y}(1 + x + y),\ e^{x-y}(1 - x - y)).$

$\begin{cases} P(x,y) = e^{x-y}(1 + x + y) \\ Q(x,y) = e^{x-y}(1 - x - y) \end{cases} \Rightarrow \begin{cases} \frac{\partial P}{\partial y} = e^{x-y}(-x - y) \\ \frac{\partial Q}{\partial x} = e^{x-y}(-x - y) \end{cases} \Rightarrow$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (villkoret är uppfyllt) $\Rightarrow$ F är ett potentialfält.

c) $\underline{\mathbf{F}(x,y)=(y,\,xy-x)}$

$$\begin{cases} P(x,y)=y \\ Q(x,y)=x(y-1) \end{cases} \Rightarrow \begin{cases} \frac{\partial P}{\partial y}=1 \\ \frac{\partial Q}{\partial x}=y-1 \end{cases} \Rightarrow \frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x} \Rightarrow \underline{\mathbf{F}\text{ är}}$$

$\underline{\text{inget potentialfält.}}$

d) $\underline{\mathbf{F}(x,y,z)=(y^2\cos x+z^3,\ 2y\sin x-4,\ 3xz^2+2)}$

$$\operatorname{rot}\mathbf{F}(x,y,z)=\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2+2 \end{vmatrix}=$$

$=(0,0,0) \Rightarrow \underline{\mathbf{F}\text{ är ett potentialfält.}}$

e) $\underline{\mathbf{F}(x,y,z)=(y^2z+\frac{1}{3}z^3,\ 2xyz+yz,\ xy^2+xz^2+\frac{1}{2}y^2).}$

$$\operatorname{rot}\mathbf{F}=\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2z+z^3/3 & 2xyz+yz & xy^2+xz^2+y^2/2 \end{vmatrix}=$$

$=(2xy+y-2xy-y,\ y^2+z^2-y^2-z^2,\ 2yz-2yz)=$

$=(0,0,0)=0 \Rightarrow \underline{\mathbf{F}\text{ är ett potentialfält.}}$

f) $\underline{\mathbf{F}(x,y,z)=(2xy^3,\ x^2z^3,\ 3x^2yz^2).}$

$\operatorname{rot}\mathbf{F}=(3x^2z^2-3x^2z^2,\ -6xyz^2,\ 2xz^3-6xy^2)=$

$=(0,-6xyz^2,\ 2xz^3-6xy^2)\neq 0 \Rightarrow \underline{\mathbf{F}\text{ är inget potentialfält.}}$

Uppgift 8.2 (Sid. 10)

Lösning

$\mathbf{A}(x,y)=\left(\frac{axy}{1+x^2y},\ \frac{x^2}{1+x^2y}\right)$ potentialfält $\Leftrightarrow$

$\operatorname{rot}\mathbf{A}=0 \Leftrightarrow \frac{\partial}{\partial x}\left(\frac{x^2}{1+x^2y}\right)=\frac{\partial}{\partial y}\left(\frac{axy}{1+x^2y}\right) \Leftrightarrow$

$\Leftrightarrow \frac{2x}{1+x^2y}-\frac{4x^3y}{(1+x^2y)^2}=\frac{ax}{1+x^2y}-\frac{2ax^3y}{(1+x^2y)^2} \Leftrightarrow$

$\Leftrightarrow \underline{a=2.}$

a) $\operatorname{grad}U(x,y)=\mathbf{A}(x,y) \Leftrightarrow \begin{cases} \frac{\partial U}{\partial x}=\frac{2xy}{1+x^2y} & (1) \\ \frac{\partial U}{\partial y}=\frac{x^2}{1+x^2y} & (2) \end{cases}$;

$(1) \Rightarrow U(x,y)=\ln(1+x^2y)+f(y) \Rightarrow \frac{\partial U}{\partial y}=\frac{x^2}{1+x^2y}+$

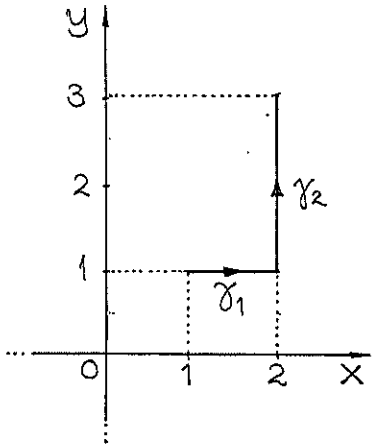
$+f'(y) \overset{(2)}{=} \frac{x^2}{1+x^2y} \Rightarrow f'(y)=0 \Leftrightarrow f(y)=C.$

$\underline{U(x,y)=\ln(1+x^2y)}$ är en skalär potential.

$\int_\gamma \mathbf{A}\cdot d\mathbf{r}=\left[\ln(1+x^2y)\right]_{(1,1)}^{(2,3)}=\ln 13-\ln 2=\underline{\ln\frac{13}{2}}.$

b) Båda punktena ligger i det övre halv-

planet och där är $\mathbb{A}$ kontinuerligt.



$\gamma = \gamma_1 + \gamma_2$

$$\int_\gamma \mathbb{A}\cdot d\mathbf{r} = \int_{\gamma_1} \mathbb{A}\cdot d\mathbf{r} + \int_{\gamma_2} \mathbb{A}\cdot d\mathbf{r} =$$
$$= \int_1^2 \frac{2x}{1+x^2}dx + \int_1^3 \frac{4}{1+4y}dy =$$
$$= \left[\ln(1+x^2)\right]_1^2 + \left[\ln(1+4y)\right]_1^3 =$$
$$= \ln 5 - \ln 2 + \ln 13 - \ln 5 =$$
$$= \ln 13 - \ln 2 = \underline{\ln\frac{13}{2}}.$$

Anm. På γ_1 är $dy=0$ och på γ_2 är $dx=0$.

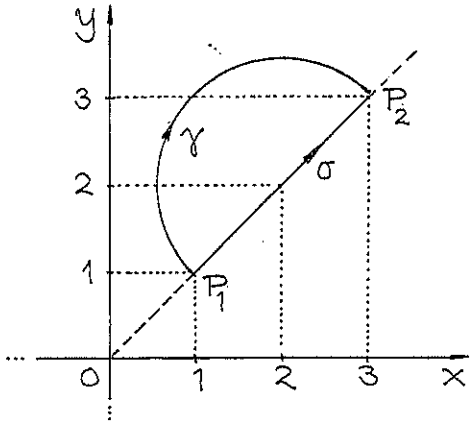
Uppgift 8.3 (Sid. 10)

Lösning

$\mathbb{A}(x,y) = \left(\frac{1}{x}\arctan\frac{y}{x}, \frac{1}{y}\arctan\frac{y}{x}\right) \in \mathcal{C}^1$ i $y>0$.

$$\begin{cases} P(x,y) = \frac{1}{x}\operatorname{atn}\frac{y}{x} \\ Q(x,y) = \frac{1}{y}\operatorname{atn}\frac{x}{y} \end{cases} \Rightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{1}{x^2+y^2} \\ \frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2} \end{cases} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$$

$\Rightarrow \mathbb{A}(x,y)$ är ett potentialfält så $\int_\gamma \mathbb{A}\cdot d\mathbf{r}$ är oberoende av γ. Jag väljer en ... genväg.



$\sigma: \mathbf{r}(t) = (t,t),\ 0 \le t \le 1$

$$\int_\gamma \mathbb{A}\cdot d\mathbf{r} = \int_\sigma \mathbb{A}\cdot d\mathbf{r} = \int_1^3 \mathbb{A}(t,t)\cdot(1,1)\,dt =$$
$$= \int_1^3 \left(\frac{1}{t}\operatorname{atn}1, \frac{1}{t}\operatorname{atn}1\right)\cdot(1,1)\,dt =$$
$$= \int_1^3 2\cdot\operatorname{atn}1\cdot\frac{dt}{t} = 2\cdot\frac{\pi}{4}\left[\ln t\right]_1^3 =$$
$$= \underline{\frac{\pi\ln 3}{2}}.$$

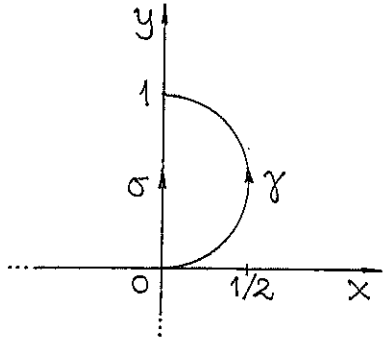
Anm. $\operatorname{atn}x = \arctan x = \tan^{-1}x$.

Uppgift 8.4 (Sid. 10)

Lösning

$$A(x,y)=(2x\cdot \operatorname{atn} y, \frac{x^2+1}{y^2+1}) \Leftrightarrow \begin{cases} P(x,y)=2x\operatorname{atn} y \\ Q(x,y)=\frac{x^2+1}{y^2+1} \end{cases} \Rightarrow$$

$\frac{\partial P}{\partial y}=\frac{2x}{y^2+1}=\frac{\partial Q}{\partial x} \Rightarrow$ integralen oberoende av vägen.



$\sigma: r(t)=(0,t),\ 0\le t\le 1$

$$\int_\gamma A\cdot dr=\int_\sigma A\cdot dr=\int_0^1 (0,\frac{1}{t^2+1})\cdot(0,1)\,dt=\int_0^1 \frac{dt}{t^2+1}=$$
$$=[\arctan t]_0^1=\operatorname{atn} 1=\frac{\pi}{4}.$$

Uppgift 8.5 (Sid. 10)

Lösning

$$\mathcal{F}(x,y)=\frac{1}{(x^2+y^2)^2}(y^2-x^2,-2xy).$$

$$P(x,y)=\frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow \frac{\partial P}{\partial y}=\frac{2y}{(x^2+y^2)^2}-\frac{4y(y^2-x^2)}{(x^2+y^2)^3}=$$
$$=\frac{2y(x^2+y^2)-4y(y^2-x^2)}{(x^2+y^2)^3}=\frac{2x^2y+2y^3-4y^3+4x^2y}{(x^2+y^2)^3}=\frac{6x^2y-2y^3}{(x^2+y^2)^3}.$$

$$Q(x,y)=-\frac{-2xy}{(x^2+y^2)^2} \Rightarrow \frac{\partial Q}{\partial x}=\frac{-2y}{(x^2+y^2)^2}+\frac{8x^2y}{(x^2+y^2)^3}=$$
$$=\frac{8x^2y-2y(x^2+y^2)}{(x^2+y^2)^3}=\frac{8x^2y-2x^2y-2y^3}{(x^2+y^2)^3}=\frac{6x^2y-2y^3}{(x^2+y^2)^3}=$$
$$=\frac{\partial P}{\partial y} \Rightarrow \mathcal{F}$$ potentialfält i området $x+y>0$

$\Rightarrow$ arbetsintegralen oberoende av vägen.

$$\operatorname{grad} U=(P,Q) \Leftrightarrow \begin{cases} \frac{\partial U}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2} & (1) \\ \frac{\partial U}{\partial y}=-\frac{2xy}{(x^2+y^2)^2} & (2) \end{cases};$$

$$(2) \Rightarrow U(x,y)=\frac{x}{x^2+y^2}+f(x) \Rightarrow \frac{\partial U}{\partial x}=\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}+$$
$$+f'(x)=\frac{y^2-x^2}{(x^2+y^2)^2}+f'(x) \overset{(1)}{=} \frac{y^2-x^2}{(x^2+y^2)^2} \Leftrightarrow f'(x)=0 \Leftrightarrow f(x)=C;$$

$U(x,y)=\frac{x}{x^2+y^2}$ är en skalärpotential till $\mathcal{F}$.

$$\int_\gamma \mathcal{F}\cdot dr=U(1,0)-U(0,3)=\underline{1}$$

Uppgift 8.6 (Sid. 10)

Lösning

$$A(x,y)=((a+2x^2+2xy)e^{x^2+4y^2},(a+8xy+8y^2)e^{x^2+4y^2}).$$

a) $P(x,y)=(a+2x^2+2xy)e^{x^2+4y^2} \Rightarrow \frac{\partial P}{\partial y}=2xe^{x^2+4y^2}+$

$+8y(a+2x^2+2xy)e^{x^2+4y^2} = (2x+8ay+16x^2y+16xy^2)e^{x^2+4y^2}$.

$Q(x,y) = (a+8xy+8y^2)e^{x^2+4y^2} \Rightarrow \frac{\partial Q}{\partial x} = 8ye^{x^2+4y^2} +$

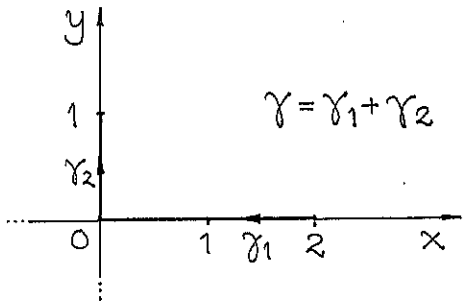
$+2x(a+8xy+8y^2)e^{x^2+4y^2} = (2ax+8y+16x^2y+16xy^2)e^{x^2+4y^2}$.

$\int_\gamma \mathbb{A}\cdot d\mathbf{r}$ oberoende av $\gamma \Leftrightarrow \mathbb{A}$ potentialfält $\Leftrightarrow$

$\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow 2x+8ay+16x^2y+16xy^2 = 2ax+8y+$

$+16x^2y+16xy^2 \Leftrightarrow \underline{a=1}$.

b)



$\mathbb{A}(x,y) = ((1+2x^2+2xy)e^{x^2+4y^2}, (1+8xy+8y^2)e^{x^2+4y^2})$.

$\int_\gamma \mathbb{A}\cdot d\mathbf{r} = \int_{\gamma_1} \mathbb{A}\cdot d\mathbf{r} + \int_{\gamma_2} \mathbb{A}\cdot d\mathbf{r} = \int_2^0 \mathbb{A}(x,0)\cdot(1,0)\,dx +$

$+ \int_0^1 \mathbb{A}(0,y)\cdot(0,1)\,dy =$

$= \int_2^0 (1+2x^2)e^{x^2}dx + \int_0^1 (1+8y^2)e^{4y^2}dy =$

$= [xe^{x^2}]_2^0 + [ye^{4y^2}]_0^1 = -2e^4 + e^4 = \underline{-e^4}$.

Svar: a) $a=1$; b) $-e^4$.

Uppgift 8.7 (Sid. 10)

Lösning

a) $\mathbb{A}(x,y) = \left(\frac{y^2}{1+xy^2}, \frac{2xy}{1+xy^2}\right) \Rightarrow \begin{cases} P(x,y) = \frac{y^2}{1+xy^2} \\ Q(x,y) = \frac{2xy}{1+xy^2} \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{2y}{1+xy^2} - \frac{2xy^3}{(1+xy^2)^2} = \frac{2y}{(1+xy^2)^2} \\ \frac{\partial Q}{\partial x} = \frac{2y}{1+xy^2} - \frac{2xy^3}{(1+xy^2)^2} = \frac{2y}{(1+xy^2)^2} \end{cases} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$

$\Rightarrow \mathbb{A}(x,y)$ har skalärpotential $\Rightarrow \int_\gamma \mathbb{A}\cdot d\mathbf{r}$ är oberoende av γ.

b) $\mathbb{A}\cdot d\mathbf{r} = \frac{y^2dx+2xydy}{1+xy^2} = \frac{d(xy^2)}{1+xy^2} = d(\ln(1+xy^2)) \Rightarrow$

$\int_\gamma \mathbb{A}\cdot d\mathbf{r} = [\ln(1+xy^2)]_{(1,1)}^{(a,b)} = \ln\frac{1+ab^2}{2} = 0 \Leftrightarrow$

$\Leftrightarrow \frac{1+ab^2}{2} = 1 \Leftrightarrow \underline{ab^2=1}$.

De sökta punkterna uppfyller $xy^2=1$.

Uppgift 8.8 (Sid. 10)

Lösning

$\mathrm{rot}(f\mathbb{A}) = (\mathrm{grad} f)\times\mathbb{A} + f\,\mathrm{rot}\,\mathbb{A}$ (Se 4.6 b)).

$f\mathbb{A}$ är ett potentialfält om det är irrota-

tionellt (virvelfritt), dvs om $\operatorname{rot}(f\mathbb{A})=0$.

$$\operatorname{rot}\mathbb{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ 3yz & 3xz & xy\end{vmatrix}=(-2x,2y,0).$$

$$\operatorname{grad}f\times\mathbb{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x f & \partial_y f & 0\\ 3yz & 3xz & xy\end{vmatrix}=$$

$$=\left(xy\frac{\partial f}{\partial y},-xy\frac{\partial f}{\partial x},3xz\frac{\partial f}{\partial x}-3yz\frac{\partial f}{\partial y}\right).$$

$$\left(xy\frac{\partial f}{\partial y},-xy\frac{\partial f}{\partial x},3xz\frac{\partial f}{\partial x}-3yz\frac{\partial f}{\partial y}\right)+f(x,y)(-2x,2y,0)=$$

$$=\left(xy\frac{\partial f}{\partial y}-2xf(x,y),-xy\frac{\partial f}{\partial x}+2yf(x,y),3z\left(x\frac{\partial f}{\partial x}-y\frac{\partial f}{\partial y}\right)\right)=\underline{0}$$

$$\Leftrightarrow\begin{cases}y\frac{\partial f}{\partial y}=2f\\ x\frac{\partial f}{\partial x}=2f\\ x\frac{\partial f}{\partial x}=y\frac{\partial f}{\partial y}\end{cases}\Leftrightarrow\begin{cases}x\frac{\partial f}{\partial x}=2f\\ y\frac{\partial f}{\partial y}=2f\\ f(x,y)=g(xy)\end{cases}\Leftrightarrow\begin{cases}f(x,y)=g(t)\\ t=xy\\ tg'(t)=2g(t)\end{cases}\Leftrightarrow$$

$$\Leftrightarrow f(x,y)=g(t)=Ct^2=C(xy)^2=\underline{Cx^2y^2}.$$

Uppgift 8.9 (Sid. 10)

Lösning

a) $\mathbb{A}(x,y,z)=(axy+z^3,x^2,bxz^2)\Rightarrow\operatorname{rot}\mathbb{A}(x,y,z)=$

$$=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ axy+z^3 & x^2 & bxz^2\end{vmatrix}=(0,3z^2-bz^2,2x-ax);$$

A potentialfält $\Rightarrow\operatorname{rot}\mathbb{A}=0\Leftrightarrow\underline{a=2}\wedge\underline{b=3}$.

b) $\operatorname{grad}f(x,y,z)=\mathbb{A}(x,y,z)=(2xy+z^3,x^2,3xz^2)\Leftrightarrow$

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=(2xy+z^3)dx+x^2dy+$$

$$+3xz^2dz=(2xy\,dx+x^2dy)+(z^3dx+3xz^2dz)=$$

$$=d(x^2y)+d(xz^3)=d(x^2y+xz^3);$$

En potential till fältet är alltså

$$\underline{f(x,y,z)=x^2y+xz^3}.$$

c) $\int_\gamma\mathbb{A}\cdot dr=f(3,1,4)-f(1,-2,1)=9+192-(-2+1)=\underline{202}.$

Uppgift 8.10 (Sid. 10)

Lösning

$$\omega=(e^{-y}-ze^{-x})dx+(e^{-z}-xe^{-y})dy+(e^{-x}-ye^{-z})dz=$$

$$=(e^{-y}dx-xe^{-y}dy)+(e^{-x}dz-ze^{-x}dx)+(e^{-z}dy-$$

$$-ye^{-z}dz)=\underbrace{d(xe^{-y})+d(ze^{-x})+d(ye^{-z})}_{\text{(”gissning”)}}=$$

$= d(xe^{-y}+ze^{-x}+ye^{-z}) \Leftrightarrow \underline{f(x,y,z)=xe^{-y}+ze^{-x}+ye^{-z}+C}$

$$\begin{cases} x(0)=0 \\ y(0)=0 \\ z(0)=0 \end{cases} \wedge \begin{cases} x(1)=1 \\ y(1)=1 \\ z(1)=1 \end{cases} \Rightarrow \int_\gamma \omega = f(1,1,1)-f(0,0,0) = \underline{\frac{3}{e}}.$$

Anm. Krångliga differentialformer och/eller integrationsvägar skvallrar om exaktheten hos formen, dvs vägberoende.

$\mathbb{A}=(e^{-y}-ze^{-x},\ e^{-z}-xe^{-y},\ e^{-x}-ye^{-z}) \Rightarrow \operatorname{rot}\mathbb{A}=0 \Rightarrow$

$\mathbb{A}$ potentialfält $\Rightarrow \operatorname{grad} f(\mathbb{x}) = \mathbb{A}$, för ngt $f \Rightarrow$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = e^{-y}-ze^{-x} & (1) \\ \frac{\partial f}{\partial y} = e^{-z}-xe^{-y} & (2) \\ \frac{\partial f}{\partial z} = e^{-x}-ye^{-z} & (3) \end{cases}$$

$(1) \Rightarrow \underline{f(x,y,z)=xe^{-y}+ze^{-x}+\phi(y,z)} \Rightarrow \frac{\partial f}{\partial y} = -xe^{-y} +$

$+\frac{\partial \phi}{\partial y} \overset{(2)}{=} e^{-z}-xe^{-y} \Leftrightarrow \frac{\partial \phi}{\partial y} = e^{-z} \Leftrightarrow \phi(y,z) = ye^{-z}+\psi(z) \Rightarrow$

$\Rightarrow \underline{f(x,y,z)=xe^{-y}+ze^{-x}+ye^{-z}+\psi(z)} \Rightarrow \frac{\partial f}{\partial z} = e^{-x} -$

$-ye^{-z}+\psi'(z) \overset{(3)}{=} e^{-x}-ye^{-z} \Leftrightarrow \psi'(z)=0 \Leftrightarrow \psi(z)=C \Rightarrow$

$$\Rightarrow \underline{f(x,y,z)=xe^{-y}+ye^{-z}+ze^{-x}}.$$

$$\int_\gamma \mathbb{A}\cdot d\mathbb{r} = f(x(1),y(1),z(1)) - f(x(0),y(0),z(0)) =$$

$$= f(1,1,1) - f(0,0,0) = f(1,1,1) = \underline{3e^{-1}}.$$

Uppgift 8.11 (Sid. 10)

Lösning

$$\begin{cases} \mathbb{A}(x,y,z) = (y^2z+\frac{az^3}{3},\ 2xyz,\ xy^2+xz^2). \\ \gamma : \mathbb{r}(t) = (\sin t,\ 2\cos t,\ 3\sin t+\cos t),\ 0\le t\le \frac{\pi}{2} \end{cases}$$

a) $$\operatorname{rot}\mathbb{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2z+az^3/3 & 2xyz & xy^2+xz^2 \end{vmatrix} = (0,0,0)$$

$\Leftrightarrow (2xy-2xy,\ y^2+az^2-y^2-z^2,\ 2yz-2yz) = 0 \Leftrightarrow$

$\Leftrightarrow (0,\ (a-1)z^2,\ 0) = 0 \Leftrightarrow \underline{a=1}.$

b) $\operatorname{grad} f(\mathbb{x}) = \mathbb{A}(\mathbb{x}) = (y^2z+\frac{1}{3}z^3,\ 2xyz,\ xy^2+xz^2) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \frac{\partial f}{\partial x} = y^2z+\frac{1}{3}z^3 & (1) \\ \frac{\partial f}{\partial y} = 2xyz & (2) \\ \frac{\partial f}{\partial z} = xy^2+xz^2 & (3) \end{cases}$$

Se 8.10 Anm.

$(1) \Rightarrow f(x,y,z) = xy^2z+\frac{1}{3}xz^3+\phi(y,z) \Rightarrow \frac{\partial f}{\partial y} = 2xyz +$

$+\frac{\partial\phi}{\partial y} \overset{(2)}{=} 2xyz \Leftrightarrow \frac{\partial\phi}{\partial y}=0 \Leftrightarrow \phi(y,z)=\psi(z) \Rightarrow \underline{f(x,y,z)=}$

$=\underline{xy^2z+\frac{1}{3}xz^3+\psi(z)} \Rightarrow \frac{\partial f}{\partial z}=xy^2+xz^2+\psi'(z)=(3)=$

$=xy^2+xz^2 \Leftrightarrow \psi'(z)=0 \Leftrightarrow \psi(z)=C.$

$$\underline{f(x,y,z)=xy^2z+\frac{1}{3}xz^3.}$$

$$\int_\gamma A\cdot dr = f(x(\tfrac{\pi}{2}),y(\tfrac{\pi}{2}),z(\tfrac{\pi}{2}))-f(x(0),y(0),z(0))=$$

$$=f(1,0,3)-f(0,2,1)=9-0=\underline{9.}$$

Resultat: a) $a=1$; b) $\int_\gamma A\cdot dr=9.$

Uppgift 8.12 (Sid. 11)

Lösning

$$\underline{A(x,y,z)=(2xy+f(y,z),\ x^2+2yz+2x+z,\ y^2+2xz+y)}$$

a) $$\operatorname{rot} A=0 \Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 2xy+f(y,z) & x^2+2yz+2x+z & y^2+2xz+y \end{vmatrix} =$$

$$=(2y+1-2y-1,\ \frac{\partial f}{\partial z}-2z,\ 2x+2-2x-\frac{\partial f}{\partial y})=(0,0,0)\Leftrightarrow$$

$$\Leftrightarrow \frac{\partial f}{\partial z}=2z \wedge \frac{\partial f}{\partial y}=2 \Leftrightarrow f(y,z)=z^2+\phi(y)=2y+\psi(z)$$

$$\Rightarrow \phi(y)=2y \wedge \psi(z)=z^2 \Rightarrow \underline{f(y,z)=z^2+2y}$$

b) $\operatorname{grad} U(x)=(2xy+2y+z^2,\ x^2+2yz+2x+z,\ y^2+2xz+y)$

$$\Leftrightarrow \begin{cases} \frac{\partial U}{\partial x}=2xy+2y+z^2 & (1) \\ \frac{\partial U}{\partial y}=x^2+2yz+2x+z & (2) \\ \frac{\partial U}{\partial z}=y^2+2xz+y & (3) \end{cases}$$

$(1)\Rightarrow \underline{U(x)=x^2y+2xy+xz^2+\phi(y,z)} \Rightarrow \frac{\partial U}{\partial y}=x^2+$

$+2x+\frac{\partial\phi}{\partial y} \overset{(2)}{=} x^2+2yz+2x+z \Leftrightarrow \frac{\partial\phi}{\partial y}=2yz+z \Leftrightarrow$

$\phi(y,z)=y^2z+yz+\psi(z) \Rightarrow \underline{U(x)=x^2y+2xy+xz^2+}$

$\underline{+y^2z+yz+\psi(z)} \Rightarrow \frac{\partial U}{\partial z}=2xz+y^2+y+\psi'(z) \overset{(3)}{=} y^2+$

$+2xz+y \Leftrightarrow \psi'(z)=0 \Leftrightarrow \underline{\psi(z)=C.}$

$$\underline{U(x,y,z)=x^2y+2xy+xz^2+y^2z+yz.}$$

c) $$\int_\gamma A\cdot dr = f(x(0),y(0),z(0))-f(x(\tfrac{3\pi}{2}),y(\tfrac{3\pi}{2}),z(\tfrac{3\pi}{2}))$$

$$=f(0,1,1)-f(-1,0,-1)=2-(-1)=\underline{3.}$$

Uppgift 8.13 (Sid. 11)

Lösning

$$A(x)=(P(x),Q(x),R(x)) \Leftrightarrow \begin{cases} P(x)=e^x\sin y\cos z \\ Q(x)=e^x\cos y\cos z \\ R(x)=-e^x\sin y\sin z \end{cases};$$

a) $\text{rot}\,\mathbb{A} = (\partial_y R - \partial_z Q,\ \partial_z P - \partial_x R,\ \partial_x Q - \partial_y P) =$

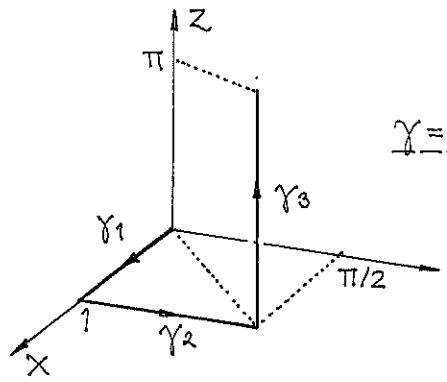
$= (-e^x \cos y \sin z + e^x \cos y \sin z)\,\hat{x} +$

$+ (-e^x \sin y \sin z + e^x \sin y \sin z)\,\hat{y} +$

$+ (e^x \cos y \cos z - e^x \cos y \cos z)\,\hat{z} = 0$

$\Rightarrow$ $\mathbb{A}$ är ett potentialfält $\Rightarrow \int_\gamma \mathbb{A}\cdot d\mathbf{r}$ <u>oberoende av γ</u> mellan $(0,0,0)$ och $(1,\pi/2,\pi)$

b)



$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

$$\int_\gamma \mathbb{A}\cdot d\mathbf{r} = \int_{\gamma_1} \mathbb{A}\cdot d\mathbf{r} + \int_{\gamma_2} \mathbb{A}\cdot d\mathbf{r} + \int_{\gamma_3} \mathbb{A}\cdot d\mathbf{r} =$$

$$= \int_0^1 P(x,0,0)\,dx + \int_0^{\pi/2} Q(1,y,0)\,dy +$$

$$+ \int_0^{\pi} R\left(1,\tfrac{\pi}{2},z\right)dz =$$

$$= \int_0^1 0\,dx + \int_0^{\pi/2} e\cos y\,dy - \int_0^{\pi} e\sin z\,dz =$$

$$= 0 + e\sin\tfrac{\pi}{2} + e(\cos\pi - 1) = e - 2e = \underline{-e}.$$

Uppgift 8.14 (Sid. 11)

Lösning

$\underline{\mathbb{A}(x,y,z) = (4xy+3x^2z^2,\ f(x)+2yz,\ y^2+2x^3z+2z)}$

a) $\mathbb{A}$ är ett ett potentialfält om och endast om det är irrotationellt, dvs om $\text{rot}\,\mathbb{A} = 0$.

$$\text{rot}\,\mathbb{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 4xy+3x^2z^2 & f(x)+2yz & y^2+2x^3z+2z \end{vmatrix} =$$

$$= (2y-2y,\ 6x^2z-6x^2z,\ f'(x)-4x) = (0,0,0)$$

$\Leftrightarrow f'(x) = 4x \Rightarrow \underline{f(x) = 2x^2 + C}.$

b) $\text{grad}\,U = \mathbb{A} = (4xy+3x^2z^2,\ 2x^2+2yz,\ y^2+2x^3z+2z)$

$$\Leftrightarrow \begin{cases} \dfrac{\partial U}{\partial x} = 4xy + 3x^2z^2 & (1) \\ \dfrac{\partial U}{\partial y} = 2x^2 + 2yz + C & (2) \\ \dfrac{\partial U}{\partial z} = y^2 + 2x^3z + 2z & (3) \end{cases}$$

(1) $\Rightarrow \underline{U(x) = 2x^2y + x^3z^2 + \phi(y,z)} \Rightarrow \dfrac{\partial U}{\partial y} = 2x^2 + \dfrac{\partial \phi}{\partial y} =$

$= (2) = 2x^2 + 2yz + C_1 \Leftrightarrow \dfrac{\partial \phi}{\partial y} = 2yz + C_1 \Leftrightarrow \phi(y,z) = y^2z + C_1 y +$

$+ \psi(z) \Rightarrow \underline{U(x,y,z) = 2x^2y + x^3z^2 + y^2z + C_1 y + \psi(z)} \Rightarrow$

$+y^2+\psi'(z) \overset{(3)}{=} y^2+2x^3z+2z \Leftrightarrow \psi'(z)=2z \Leftrightarrow \psi(z)=z^2+C_2$

$\Rightarrow \underline{U(x,y,z)=2x^2y+x^3z^2+y^2z+z^2+C_1y+z^2+C_2.}$

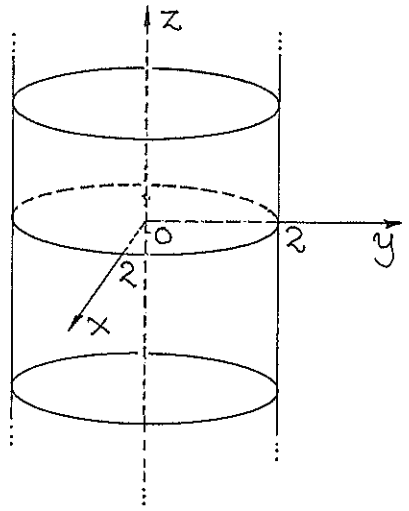
9. Kroklinjiga koordinater

Uppgift 9.1 (Sid. 11)

Lösning

a) $\underline{\rho=2}$ (Cylindriska koordinater ρ, φ, z).

En cylinderyta som i figuren nedan.

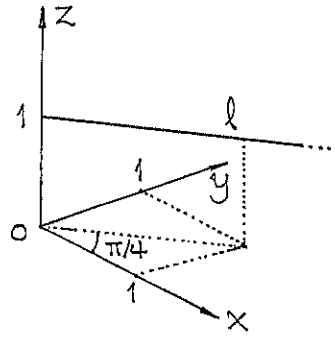


Ytan har oändlig utsträckning $(-\infty<z<\infty)$.

b) $\underline{\varphi=\frac{\pi}{4},\ z=1}$ (cylindriska koordinater).

$$\begin{cases} x=\rho\cos\varphi=\rho/\sqrt{2} \\ y=\rho\sin\varphi=\rho/\sqrt{2} \\ z=1 \end{cases} \Leftrightarrow \begin{cases} x=y\geq 0 \\ z=1 \end{cases} \Leftrightarrow \begin{cases} x=t \\ y=t, \ t\geq 0. \\ z=1 \end{cases}$$

$$\Leftrightarrow \underline{(x,y,z)=(0,0,1)+t(1,1,0),\ t\geq 0.}$$

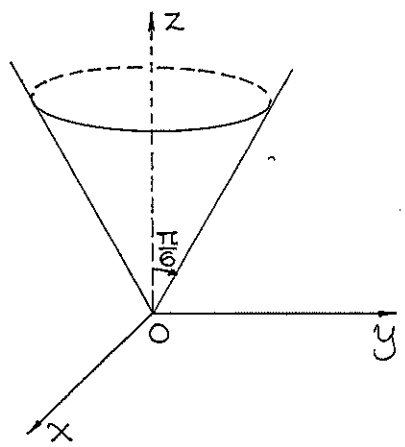


Den geometriska orten är skärningen l mellan halvplanet $y=x\geq 0$ och planet $z=1$.

c) $\underline{\theta=\frac{\pi}{6}}$ (rymdpolära koordinater r,θ,φ).

$$\begin{cases} x=r\sin(\pi/6)\cos\varphi \\ y=r\sin(\pi/6)\sin\varphi \\ z=\cos(\pi/6)\cdot r=r/2 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2=r^2 \\ r=2z\geq 0 \end{cases} \Rightarrow x^2+y^2=4z^2$$

$\Leftrightarrow z=\frac{1}{2}\sqrt{x^2+y^2}$, en konisk yta med spetsen i origo och symmetriaxeln den positiva z-axeln; spetsvinkeln är $\frac{\pi}{3}$ (se figur).

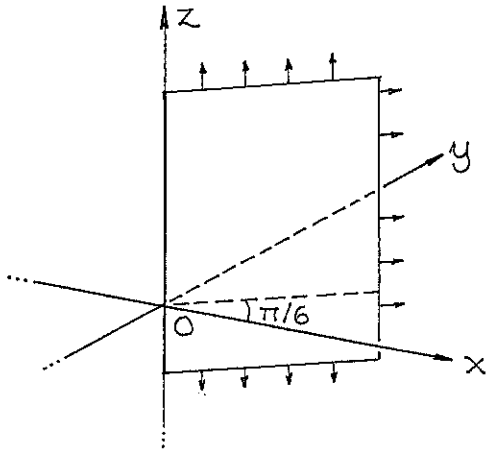


Konen är rotationssymmetrisk och har oändlig utsträckning ($0 \leq z < \infty \Leftrightarrow 0 < r < \infty$).

d) $\varphi = \frac{\pi}{6}$ (cylindriska koordinater ρ, φ, z)

$x = \rho\cos\frac{\pi}{6} \wedge y = \rho\sin\frac{\pi}{6} \Rightarrow \frac{y}{x} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \Rightarrow x = \sqrt{3}y$;

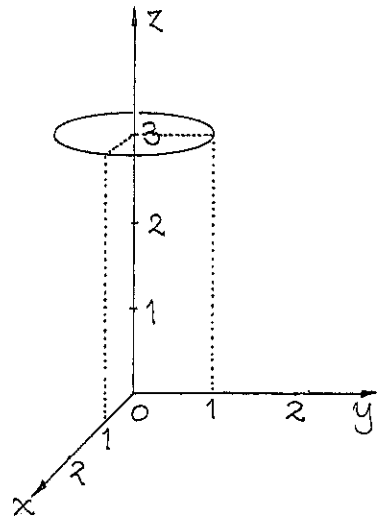
ett halvplan som i figuren nedan.



Anm. Även i rymdpolära koordinater r, θ, φ är grafen exakt samma plan

e) $\rho = 1, z = 3$ (cylindriska koordinater)

$\begin{cases} \rho = \sqrt{x^2+y^2} = 1 \\ z = 3 \end{cases}$; enhetscirkeln i planet $z=3$.

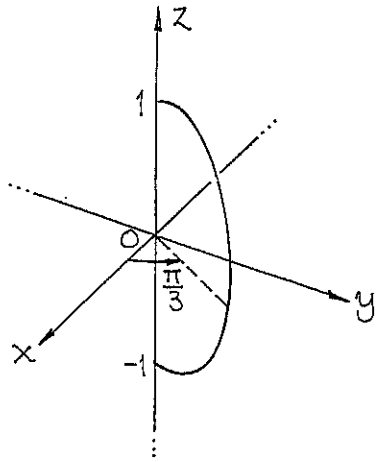


Cirkeln tolkas som skärningen mellan cylindern $x^2+y^2=1$ och planet $z=3$.

f) $r = 1, 0 \leq \theta \leq \pi, \varphi = \pi/3$ (rymdpolära)

$x = \sin\theta \cdot \frac{1}{2}$, $y = \sin\theta \cdot \frac{\sqrt{3}}{2}$ och $z = \cos\theta$;

$y = \sqrt{3}x \wedge x^2+y^2+z^2 = 1$, $x, y \geq 0$.



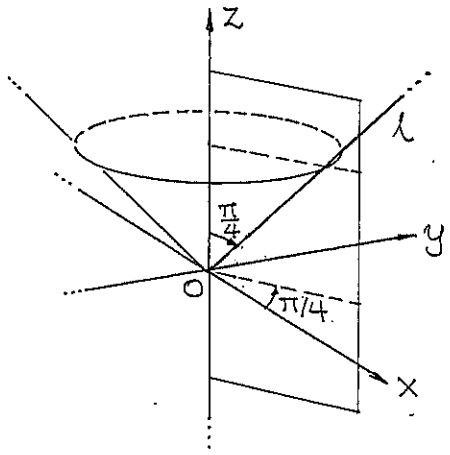
Det är frågan om en halv storcirkel till sfären $x^2+y^2+z^2=1$ (enhetssfären) i halvplanet $\sqrt{3}x-y=0$, $x,y\geq 0$.

c) $r\geq 0$, $\theta=\frac{\pi}{4}$, $\varphi=\frac{\pi}{4}$ (rymdpolära koordinater).

$$\begin{cases} x=r\sin\theta\cos\varphi=r/2 \\ y=r\sin\theta\sin\varphi=r/2 \\ z=r\cos\theta=r/\sqrt{2}=(r/2)\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} x=t \\ y=t \\ z=\sqrt{2}t \end{cases}, \quad t\geq 0.$$

En stråle utgående från origo med riktningsvektor(n) $v=(1,1,\sqrt{2})$.

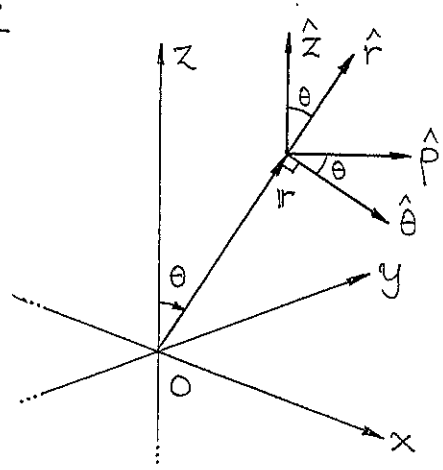
Anm. $\theta=\frac{\pi}{4}$ motsvarar konen $z=\sqrt{x^2+y^2}$;
$\varphi=\frac{\pi}{4}$ motsvarar halvplanet $x-y=0$; $x,y\geq 0$.



l är skärningen mellan den koniska ytan och halvplanet.

Uppgift 9.2 (Sid. 11)

Lösning



a) $\hat{\rho}=(\hat{\rho}\cdot\hat{r})\hat{r}+(\hat{\rho}\cdot\hat{\theta})\hat{\theta}=\cos(\frac{\pi}{2}-\theta)\hat{r}+\cos\theta\,\hat{\theta}=\underline{\sin\theta\,\hat{r}+\cos\theta\,\hat{\theta}}$.

b) $\hat{z} = (\hat{z}\cdot\hat{r})\hat{r} + (\hat{z}\cdot\theta)\hat{\theta} = \cos\theta\hat{r} + \cos(\frac{\pi}{2}+\theta)\hat{\theta} = \underline{\cos\theta\hat{r} - \sin\theta\hat{\theta}}.$

Uppgift 9.3 (Sid. 11)

Lösning

$$\underline{\Psi = -\frac{\cos\theta}{r^2}}, \quad \underline{A = \frac{\sin\theta}{r^2}\hat{\varphi}}.$$

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$;

Δ. $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$

$$h_r = 1, \; h_\theta = r, \; h_\varphi = r\sin\theta$$

a) $$\nabla\Psi = \frac{1}{h_r}\frac{\partial\Psi}{\partial r}\hat{r} + \frac{1}{h_\theta}\frac{\partial\Psi}{\partial\theta}\hat{\theta} + \frac{1}{h_\varphi}\frac{\partial\Psi}{\partial\varphi}\hat{\varphi} =$$

$$= \frac{\partial\Psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{\theta} = \underline{\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}}.$$

b) $$\nabla \times A = \frac{1}{h_r h_\theta h_\varphi}\begin{vmatrix} h_r\hat{r} & h_\theta\hat{\theta} & h_\varphi\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ h_r A_r & h_\theta A_\theta & h_\varphi A_\varphi \end{vmatrix} =$$

$$= \frac{1}{1\cdot r\cdot r\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & 0 & \sin^2\theta/r \end{vmatrix} =$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{2\sin\theta\cos\theta}{r}\hat{r} + \frac{\sin^2\theta}{r^2}r\hat{\theta}\right) =$$

$$= \underline{\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}} = \nabla\Psi.$$

Uppgift 9.4 (Sid. 11)

Lösning

$$\underline{A = e^\rho\sin\varphi\,\hat{\rho} + \rho e^\rho\cos\varphi\,\hat{\varphi} + \rho^2\hat{z}}$$

$\underline{x = \rho\cos\varphi, \; y = \rho\sin\varphi, \; z = z; \quad \rho \geq 0, \; 0 \leq \varphi \leq 2\pi, \; -\infty < z < \infty.}$

$$\underline{h_\rho = 1, \; h_\varphi = \rho, \; h_z = 1}$$

$$\operatorname{div} A = \frac{1}{h_\rho h_\varphi h_z}\left(\partial_\rho(h_\varphi h_z A_\rho) + \partial_\varphi(h_\rho h_z A_\varphi) + \partial_z(h_\rho h_\varphi A_z)\right)$$

$$= \frac{1}{1\cdot\rho\cdot 1}\left(\partial_\rho(\rho e^\rho\sin\varphi) + \partial_\varphi(\rho e^\rho\cos\varphi) + \partial_z(\rho^3)\right) =$$

$$= \frac{1}{\rho}\left((\rho+1)e^\rho\sin\varphi - \rho e^\rho\sin\varphi\right) = \underline{\frac{e^\rho}{\rho}\sin\varphi}.$$

Uppgift 9.5 (Sid. 11)

Lösning: $\underline{A = (r\cos\theta, \; r\sin\theta, \; r\varphi)}$

$$\operatorname{rot} A = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ r\cos\theta & r\sin\theta & r\varphi \end{vmatrix} =$$

$$= \frac{1}{r^2\sin\theta}\left(0\,\hat{r} + \varphi\cdot r\hat{\theta} + (\sin\theta - r\sin\theta)r\sin\theta\,\hat{\varphi}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(r\varphi\hat{\theta} + (r - r^2)\sin^2\theta\,\hat{\varphi}\right) =$$

$$= \underline{\frac{\varphi}{r\sin\theta}\hat{\theta} + \left(\frac{1}{r} - 1\right)\sin\theta\,\hat{\varphi}}.$$

Uppgift 9.6 (Sid. 12)

Lösning

$$\mathbb{A}(\rho,\varphi,z) = (-\rho\cos\varphi)\hat{\rho} + \rho\sin\varphi\,\hat{\varphi} + z\cos\varphi\,\hat{z}$$

$$\operatorname{rot}\mathbb{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ -\rho\cos\varphi & \rho\sin\varphi & z\cos\varphi \end{vmatrix} =$$

$$= \frac{1}{\rho}\left((-z\sin\varphi)\hat{\rho} + 0\cdot\rho\hat{\varphi} + (1-\rho)\sin\varphi\,\hat{z}\right) =$$

$$= \frac{1}{\rho}\sin\varphi\,(-z\hat{\rho} + (1-\rho)\hat{z})$$

$\operatorname{rot}\mathbb{A} = 0 \Leftrightarrow \sin\varphi = 0 \Leftrightarrow \varphi = 0$, (VSV).

Uppgift 9.7 (Sid. 12)

Lösning

$$\mathbb{A} = \operatorname{grad}\phi = (\hat{\rho}\partial_\rho + \frac{1}{\rho}\hat{\varphi}\partial_\varphi + \hat{z}\partial_z)(\ln\cos\varphi - \ln\rho) =$$

$$= -\frac{1}{\rho}\hat{\rho} + \frac{1}{\rho}(-\tan\varphi)\hat{\varphi} = -\frac{1}{\rho}(\hat{\rho} + \tan\varphi\,\hat{\varphi}).$$

Uppgift 9.8 (Sid. 12)

Lösning

$\rho = \cos\varphi$, $z = \rho\sin\varphi$

Vinkeln mellan två ytor S_1 och S_2 i en gemensam punkt P_0 är vinkeln mellan deras tangentplan i samma punkt, alternativt vinkeln mellan deras normaler (gradienter)

I cylinderkoordinater har vi

$x = \rho\cos\varphi$, $y = \rho\sin\varphi$ och $z = z$.

$\rho = \cos\varphi \Rightarrow \rho^2 = \rho\cos\varphi \Leftrightarrow x^2 + y^2 = x$ (cylinder)

$z = \rho\sin\varphi = y \Leftrightarrow y - z = 0$ (plan)

Den gemensamma punkten är (x_0, y_0, z_0):

$$x_0 = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2},\ y_0 = \frac{1}{2},\ z_0 = \frac{1}{2}.$$

$$\left.\begin{aligned} S_1 &: f_1(\mathbb{x}) = x^2 + y^2 - x \Rightarrow \nabla f_1(\mathbb{x}) = (2x-1, 2y, 0) \\ S_2 &: f_2(\mathbb{x}) = y - z \Rightarrow \nabla f_2(\mathbb{x}) = (0, 1, -1) \end{aligned}\right\} \Rightarrow$$

$$\cos\theta = \frac{\nabla f_1(P_0)\cdot\nabla f_2(P_0)}{|\nabla f_1(P_0)||\nabla f_2(P_0)|} = \frac{(0,1,0)\cdot(0,1,-1)}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}.$$

Uppgift 9.9 (Sid. 12)

Lösning $\quad S: r = 3/(3+\cos\theta)$

$r=\frac{3}{3+\cos\theta} \Leftrightarrow 3r+r\cos\theta=3 \Leftrightarrow 3\sqrt{x^2+y^2+z^2}+z=3$;

Den givna ytan är en nivåyta till

$$u=f(x,y,z)=3\sqrt{x^2+y^2+z^2}+z.$$

$\operatorname{grad} f(\mathbf{r})=(3\frac{x}{r},3\frac{y}{r},3\frac{z}{r}+1)=3\hat{r}+\hat{z}$, ty $\mathbf{r}=r\cdot\hat{r}$

$\operatorname{grad} f(\mathbf{r})\cdot\mathbf{r}=3\mathbf{r}\cdot\hat{r}+\mathbf{r}\cdot\hat{z}=3r+z=3$;

$|\operatorname{grad} f(\mathbf{r})|^2=9(\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2})+6\frac{z}{r}+1$;

$|\operatorname{grad} f(P)|^2=6\cos\theta+10 \Leftrightarrow \underline{|\operatorname{grad} f(P)|=\sqrt{6\cos\theta+10}}$;

$|\mathbf{r}|=r=\frac{3}{3+\cos\theta}=|\overrightarrow{OP}|$.

$\cos\alpha=\frac{\nabla f(P)\cdot\mathbf{r}}{|\nabla f(P)|r}=\frac{3/r}{\sqrt{6\cos\theta+10}}=\frac{\cos\theta+3}{\sqrt{6\cos\theta+10}}=\cos(\mathbf{r},\nabla f)$.

Svar: Den sökta vinkeln är $\arccos\frac{3+\cos\theta}{\sqrt{6\cos\theta+10}}$.

Uppgift 9.10 (Sid. 12)

Lösning

$$\begin{cases} r=2\sin2\theta \\ \varphi=\frac{\pi}{4} \end{cases} \Rightarrow \begin{cases} x=r\sin\theta\cos\varphi=\sqrt{2}\sin2\theta\sin\theta \\ y=r\sin\theta\sin\varphi=\sqrt{2}\sin2\theta\sin\theta \\ z=2\sin2\theta\cos\theta \end{cases} \Leftrightarrow$$

$\underline{\gamma:\mathbf{r}(\theta)=(2\sqrt{2}\sin^2\theta\cos\theta,\ 2\sqrt{2}\sin^2\theta\cos\theta,\ 4\cos^2\theta\sin\theta)}$.

$\frac{d\mathbf{r}}{d\theta}=(4\sqrt{2}\sin\theta\cos^2\theta-2\sqrt{2}\sin^3\theta)\hat{x}+$

$+(4\sqrt{2}\sin\theta\cos^2\theta-2\sqrt{2}\sin^3\theta)\hat{y}+$

$+(4\cos^3\theta-8\sin^2\theta\cos\theta)\hat{z} \Rightarrow$

$$\begin{cases} x=\frac{\sqrt{6}}{4} \Rightarrow 2\sqrt{2}\sin^2\theta\cos\theta=\frac{\sqrt{6}}{4} \\ z=\frac{3}{2} \Rightarrow 4\cos^2\theta\sin\theta=\frac{3}{2} \end{cases} \Leftrightarrow \begin{cases} \sin^2\theta\cos\theta=\frac{\sqrt{3}}{8} \\ \cos^2\theta\sin\theta=\frac{3}{8} \end{cases}$$

$$\Rightarrow \frac{\sin^2\theta\cos\theta}{\cos^2\theta\sin\theta}=\frac{\sqrt{3}}{3} \Leftrightarrow \tan\theta=\frac{1}{\sqrt{3}} \Rightarrow \theta=\frac{\pi}{6} \Rightarrow \begin{cases} \sin\theta=\frac{1}{2} \\ \cos\theta=\frac{\sqrt{3}}{2} \end{cases};$$

$\mathbf{r}'(\frac{\pi}{6})=(4\sqrt{2}\cdot\frac{1}{2}\cdot\frac{3}{4}-2\sqrt{2}\cdot\frac{1}{8},\ 4\sqrt{2}\cdot\frac{1}{2}\cdot\frac{3}{4}-2\sqrt{2}\cdot\frac{1}{8},\ 4\frac{\sqrt{3}}{8}-8\cdot\frac{1}{4}\cdot\frac{\sqrt{3}}{2})$

$=(\frac{3\sqrt{2}}{2}-\frac{\sqrt{2}}{4},\ \frac{3\sqrt{2}}{2}-\frac{\sqrt{2}}{4},\ \frac{\sqrt{3}}{2}-\sqrt{3})=\underline{(\frac{5\sqrt{2}}{4},\ \frac{5\sqrt{2}}{4},\ -\frac{\sqrt{3}}{2})}$

Uppgift 9.11 (Sid. 12)

Lösning

$\mathbf{v}=\hat{r}+\hat{\varphi} \Rightarrow \hat{v}=\frac{\hat{r}+\hat{\varphi}}{\sqrt{2}}$; $\quad P_0:(2,\frac{\pi}{2},\frac{\pi}{4})$

$\operatorname{grad}\phi(\mathbf{r})=(\hat{r}\partial_r+\frac{1}{r}\hat{\theta}\partial_\theta+\frac{1}{r\sin\theta}\hat{\theta}\partial_\varphi)\frac{2+\cos\theta}{r^2}=$

$=\underline{-2\frac{2+\cos\theta}{r^3}\hat{r}-\frac{\sin\theta}{r^3}}$;

$\operatorname{grad}\phi(P_0) = -2\cdot\frac{2}{2^3}\hat{r} - \frac{1}{2^3}\hat{\theta} = -\frac{1}{2}\hat{r} - \frac{1}{8}\hat{\theta}$;

$\frac{\partial\phi}{\partial v} = \operatorname{grad}\phi(P_0)\cdot\hat{v} = (-\frac{1}{2}\hat{r} - \frac{1}{8}\hat{\theta})\cdot(\frac{1}{\sqrt{2}}\hat{r} + \frac{1}{\sqrt{2}}\hat{\varphi}) = \underline{-\frac{\sqrt{2}}{4}}$.

Uppgift 9.12 (Sid. 12)

Lösning

$\operatorname{grad}\phi(r) = F(r) = (\frac{e^{-z}}{\rho^3}2\cos\varphi, \frac{e^{-z}}{\rho^3}\sin\varphi, \frac{e^{-z}}{\rho^2}\cos\varphi) \Leftrightarrow$

$\frac{\partial\phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} + \frac{\partial\phi}{\partial z}\hat{z} = \frac{2e^{-z}}{\rho^3}\cos\varphi\,\hat{\rho} + \frac{e^{-z}}{\rho^3}\sin\varphi\,\hat{\varphi} + \frac{e^{-z}}{\rho^2}\cos\varphi\,\hat{z}$

$$\Leftrightarrow \begin{cases} \frac{\partial\phi}{\partial\rho} = \frac{2}{\rho^3}\cos\varphi\cdot e^{-z} & (1) \\ \frac{\partial\phi}{\partial\varphi} = \frac{1}{\rho^2}\sin\varphi\cdot e^{-z} & (2) \\ \frac{\partial\phi}{\partial z} = \frac{1}{\rho}\cos\varphi\cdot e^{-z} & (3) \end{cases}$$

$(1) \Rightarrow \underline{\phi = -\frac{1}{\rho^2}\cos\varphi\cdot e^{-z} + f(\varphi, z)} \Rightarrow \frac{\partial\phi}{\partial\varphi} = \frac{1}{\rho^2}\sin\varphi\cdot e^{-z} +$

$+\frac{\partial f}{\partial\varphi} \overset{(2)}{=} \frac{1}{\rho^2}\sin\varphi\cdot e^{-z} \Leftrightarrow \frac{\partial f}{\partial\varphi} = 0 \Leftrightarrow \underline{f(\varphi, z) = g(z)} \Rightarrow$

$\phi(\rho,\varphi,z) = -\frac{1}{\rho^2}\cos\varphi\, e^{-z} + g(z) \Rightarrow \frac{\partial\phi}{\partial z} = \frac{1}{\rho^2}\cos\varphi\cdot e^{-z} + g'(z) =$

$\overset{(3)}{=} \frac{1}{\rho^2}\cos\varphi\cdot e^{-z} \Leftrightarrow g'(z) = 0 \Leftrightarrow \underline{g(z) = C}$ (konstant).

Resultat: F är ett potentialfält; en potential är t.ex. $\phi(\rho,\varphi,z) = -\frac{e^{-z}}{\rho^2}\cos\varphi$.

Uppgift 9.13 (Sid. 12)

Lösning $\qquad \underline{A = \rho^n\hat{\rho}}$

(1) $\operatorname{div} A = \frac{1}{\rho}\frac{d}{d\rho}\rho^{n+1} = \frac{1}{\rho}\cdot(n+1)\rho^n = (n+1)\rho^{n-1}$;

$\operatorname{grad}(\operatorname{div} A) = \operatorname{grad}((n+1)\rho^{n-1}) = (n+1)(n-1)\rho^{n-2}\hat{\rho}$;

(2) $\operatorname{rot} A = 0 \Rightarrow \operatorname{rot}(\operatorname{rot} A) = 0$;

(3) $\operatorname{grad}(\operatorname{div} A) - \operatorname{rot}(\operatorname{rot} A) = 0 \Leftrightarrow (n+1)(n-1) = 0 \Leftrightarrow \underline{n = \pm 1}$.

Anm. I cylinderkoordinater är skalfaktorerna givna av $h_\rho = h_z = 1$, $h_\varphi = \rho$ s.a.

$$\begin{cases} \operatorname{grad}\phi = \frac{\partial\phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{\partial\phi}{\partial z}\hat{z}; \\ \operatorname{div} A = \frac{1}{\rho}(\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{\partial}{\partial\varphi}A_\varphi + \frac{\partial}{\partial z}(\rho A_z)) \\ \operatorname{rot} A = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}; \quad \underline{A = A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{z}}. \end{cases}$$

Uppgift 9.14 (Sid. 12)

Lösning

$\underline{B = \operatorname{grad}\frac{\cos\theta}{r^2},\ r \neq 0}$.

I sfäriska (eg. rymdpolära) koordinater är:

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi};$$

$$\operatorname{div} A = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial\theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial\varphi}(rA_\varphi)\right].$$

$$\mathbb{B} = -2\frac{\cos\theta}{r^3}\hat{r} + \frac{1}{r}\left(-\frac{\sin\theta}{r^2}\right)\hat{\theta} = -\frac{2\cos\theta}{r^3}\hat{r} - \frac{\sin\theta}{r^3}\hat{\theta}.$$

$$\operatorname{div}\mathbb{B} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(-\frac{\sin 2\theta}{r}\right) + \frac{\partial}{\partial\theta}\left(-\frac{\sin^2\theta}{r^2}\right)\right) =$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\sin 2\theta}{r^2} - \frac{\sin 2\theta}{r^2}\right) = \frac{1}{r^2\sin\theta}\cdot 0 = \underline{0}.$$

Uppgift 9.15 (Sid. 12)

Lösning

$$\mathbb{A} = (z^2\sin^2\varphi)\hat{\rho} + (z^2\sin 2\varphi - \frac{z}{\rho}\sin\varphi)\hat{\varphi} + (\cos\varphi + 2\rho z\sin^2\varphi)\hat{z}$$

a) $$\operatorname{rot}\mathbb{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ z^2\sin^2\varphi & \rho z^2\sin 2\varphi - z\sin\varphi & \cos\varphi + 2\rho z\sin^2\varphi \end{vmatrix} =$$

$$= \frac{1}{\rho}\big((-\sin\varphi + 2\rho z\sin 2\varphi - 2\rho z\sin 2\varphi + \sin\varphi)\hat{\rho} +$$

$$+ (2z\sin^2\varphi - 2z\sin^2\varphi)\hat{\varphi} + (z^2\sin 2\varphi - z^2\sin 2\varphi)\hat{z}\big)$$

$$= \frac{1}{\rho}(0,0,0) = \frac{1}{\rho}0 = \underline{0} \Rightarrow \underline{\mathbb{A}\text{ konservativt.}}$$

b) $$\operatorname{grad}\phi(r) = \mathbb{A}(r) \Leftrightarrow \begin{cases} \frac{\partial\phi}{\partial\rho} = z^2\sin^2\varphi & (1) \\ \frac{\partial\phi}{\partial\varphi} = \rho z^2\sin 2\varphi - z\sin\varphi & (2) \\ \frac{\partial\phi}{\partial z} = \cos\varphi + 2\rho z\sin^2\varphi & (3) \end{cases}$$

Anm. $\operatorname{grad}\phi = \frac{\partial\phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} + \frac{\partial\phi}{\partial z}\hat{z}.$

$(1) \Rightarrow \phi = \rho z^2\sin^2\varphi + f(\varphi, z) \Rightarrow \frac{\partial\phi}{\partial\varphi} = \rho z^2\sin 2\varphi + \frac{\partial f}{\partial\varphi} =$

$\overset{(2)}{=} \rho z^2\sin 2\varphi - z\sin\varphi \Leftrightarrow \frac{\partial f}{\partial\varphi} = -z\sin\varphi \Rightarrow f(\varphi, z) = z\cos\varphi +$

$+ g(z) \Rightarrow \phi = \rho z^2\sin^2\varphi + z\cos\varphi + g(z) \Rightarrow \frac{\partial\phi}{\partial z} = 2\rho z\sin^2\varphi +$

$+ \cos\varphi + g'(z) \overset{(3)}{=} \cos\varphi + 2\rho z\sin^2\varphi \Leftrightarrow g'(z) = 0 \Leftrightarrow g(z) = C$

$$\Rightarrow \underline{\phi(\rho, \varphi, z) = \rho z^2\sin^2\varphi + z\cos\varphi}.$$

c) $$\int_P^Q A\cdot dr = \phi(1, \tfrac{\pi}{6}, 1) - \phi(5, \tfrac{\pi}{2}, -1) = \frac{1}{4} + \frac{\sqrt{3}}{2} - 5 = \frac{2\sqrt{3} - 19}{4}.$$

Uppgift 9.16 (Sid. 12)

Lösning

$$\mathbb{F} = \operatorname{grad}\phi = \frac{\partial\phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} + \frac{\partial\phi}{\partial z}\hat{z} = /\phi = (\frac{a}{\rho} + b\rho)\sin\varphi/ =$$

$$= (-\frac{a}{\rho^2} + b)\sin\varphi\,\hat{\rho} + (\frac{a}{\rho} + b\rho)\frac{1}{\rho}\cos\varphi\,\hat{\varphi} =$$

$$= (b - \frac{a}{\rho^2})\sin\varphi\,\hat{\rho} + (\frac{a}{\rho^2} + b)\cos\varphi\,\hat{\varphi};$$

Gauss' sats ger med $S=\partial D$:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\, dV = \iiint_D \nabla^2\phi\, dV \overset{!}{=} \iiint_D 0\, dV = 0.$$

Anm. $\overset{!}{=}$ underförstås följande räkning:

$$\operatorname{div}\left(\left(b-\frac{a}{\rho^2}\right)\sin\varphi\,\hat{\rho}+\left(b+\frac{a}{\rho^2}\right)\cos\varphi\,\hat{\varphi}\right) =$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho b-\frac{a}{\rho}\right)\sin\varphi+\frac{\partial}{\partial\varphi}\left(b+\frac{a}{\rho^2}\right)\cos\varphi\right) =$$

$$= \frac{1}{\rho}\left(\left(b+\frac{a}{\rho^2}\right)\sin\varphi-\left(b+\frac{a}{\rho^2}\right)\sin\varphi\right) = 0.$$

Uppgift 9.17 (Sid. 12)

Lösning

$$S:\ \mathbf{r}(\theta,\varphi) = \frac{1}{2-\cos\theta}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\Delta:\ 0\le\theta\le\pi,\ 0\le\varphi\le 2\pi$$

$$\mathbf{r}'_\theta = \left(\frac{(2\cos\theta-1)\cos\varphi}{(2-\cos\theta)^2}, \frac{(2\cos\theta-1)\sin\varphi}{(2-\cos\theta)^2}, -\frac{2\sin\theta}{(2-\cos\theta)^2}\right);$$

$$\mathbf{r}'_\varphi = \left(\frac{-\sin\theta\sin\varphi}{2-\cos\theta}, \frac{\sin\theta\cos\varphi}{2-\cos\theta}, 0\right);$$

$$\mathbf{r}'_\theta\times\mathbf{r}'_\varphi = \frac{\sin\theta}{(2-\cos\theta)^3}(2\sin\theta\cos\varphi, 2\sin\theta\sin\varphi, 2\cos\theta-1);$$

$$\mathbf{A}(\mathbf{r})\cdot(\mathbf{r}'_\theta\times\mathbf{r}'_\varphi) = \frac{\sin\theta}{(2-\cos\theta)^2}\cdot(2\sin^2\theta\cos^2\varphi+2\sin^2\theta\sin^2\varphi+$$

$$+2\cos^2\theta-\cos\theta) = \frac{\sin\theta}{(2-\cos\theta)^2}(2-\cos\theta) =$$

$$= \frac{\sin\theta}{2-\cos\theta} \Rightarrow \iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_\Delta \frac{\sin\theta}{2-\cos\theta}\, d\theta\, d\varphi = \underline{2\pi\ln 3}.$$

Anm. $\mathbf{A}(\mathbf{r}) = \frac{1}{r}\hat{\mathbf{r}} = \frac{1}{\frac{1}{2-\cos\theta}}\hat{\mathbf{r}} = (2-\cos\theta)\hat{\mathbf{r}} =$

$$= (2-\cos\theta)(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Uppgift 9.18 (Sid. 12)

Lösning

$$x^2+y^2+(z-2)^2=4 \Leftrightarrow \rho^2+(z-2)^2=4 \Leftrightarrow \rho=\sqrt{4z-z^2};$$

$S:\ \mathbf{r}(z,\varphi) = (\sqrt{4z-z^2}\cos\varphi, \sqrt{4z-z^2}\sin\varphi, z)$ (sfären)

$$\Delta:\ 0\le z\le 4,\ 0\le\varphi\le 2\pi$$

$$\left.\begin{aligned}\mathbf{r}'_z &= \left(\frac{2-z}{\sqrt{4z-z^2}}\cos\varphi, \frac{2-z}{\sqrt{4z-z^2}}\sin\varphi, 1\right)\\ \mathbf{r}'_\varphi &= (\sqrt{4z-z^2}(-\sin\varphi), \sqrt{4z-z^2}\cos\varphi, 0)\end{aligned}\right\} \Rightarrow \mathbf{r}'_z\times\mathbf{r}'_\varphi =$$

$$= (-\sqrt{4z-z^2}\cos\varphi, -\sqrt{4z-z^2}\sin\varphi, 2-z);$$

jag räknar med den utåtriktade normalen så jag byter tecken på $\mathbf{r}'_z\times\mathbf{r}'_\varphi$ och får

$$\mathbf{A}\cdot d\mathbf{S} = \mathbf{A}(\mathbf{r})\cdot(\mathbf{r}'_\varphi\times\mathbf{r}'_z)\,d\varphi\,dz = \sqrt{4z-z^2}\cdot z\frac{\rho^2-1}{\rho}\,d\varphi\,dz =$$

$$= \sqrt{4z-z^2}\cdot z\,\frac{4z^2-z^2-1}{\sqrt{4z-z^2}}\,d\varphi\,dz = (4z^2-z^3-z)\,d\varphi\,dz;$$

Anm. $\hat{\rho} = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right) = (\cos\varphi, \sin\varphi, 0)$ så att

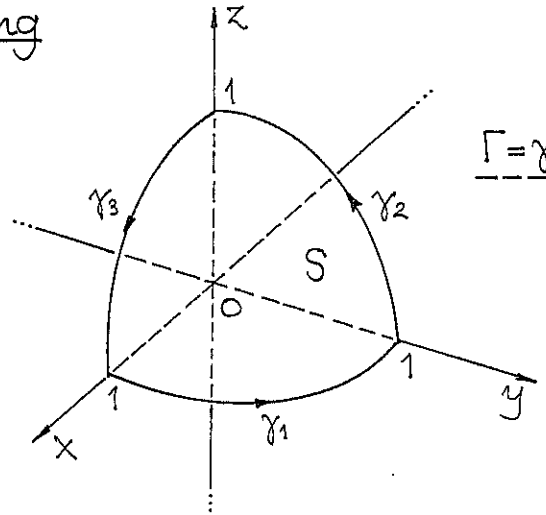
$$\mathbb{A}(\mathbf{r}) \cdot (\mathbf{r}'_\varphi \times \mathbf{r}'_z) = \sqrt{4z^2-1} \cdot z \frac{4z-z^2-1}{\sqrt{4z-z^2}} = 4z^2 - z^3 - z.$$

$$\iint_S \mathbb{A} \cdot d\mathbf{S} = \int_0^4 (4z^2 - z^3 - z)\,dz \int_0^{2\pi} d\varphi = 2\pi \left[\frac{4z^3}{3} - \frac{z^4}{4} - \frac{z^2}{2}\right]_0^4 =$$

$$= 2\pi\left(\frac{256}{12} - 8\right) = 2\pi = 2\pi\left(\frac{64}{3} - 8\right) = 2\pi \cdot \frac{40}{3} = \underline{\underline{\frac{80\pi}{3}}}.$$

Uppgift 9.19 (Sid. 13)

Lösning



$\Gamma = \gamma_1 + \gamma_2 + \gamma_3$

(1) γ_1: $d\mathbf{r} = 1 \cdot d\theta \cdot (-\hat{\theta}) = -d\theta\,\hat{\theta}$;

$$\int_{\gamma_1} \mathbb{A} \cdot d\mathbf{r} = -\int_0^{\pi/2} \sin\theta\,d\theta = \left[\cos\theta\right]_0^{\frac{\pi}{2}} = -\cos 0 = -1;$$

(2) γ_2: $d\mathbf{r} = 1 \cdot d\theta \cdot \hat{\theta} = d\theta\,\hat{\theta}$;

$$\int_{\gamma_2} \mathbb{A} \cdot d\mathbf{r} = \int_0^{\pi/2} \sin\theta\,d\theta = \left[-\cos\theta\right]_0^{\pi/2} = 1.$$

(3) γ_3: $d\mathbf{r} = 1 \cdot d\varphi\,\hat{\varphi} = d\varphi\,\hat{\varphi}$;

$$\int_{\gamma_3} \mathbb{A} \cdot d\mathbf{r} = \int_0^{\pi/2} \sin\frac{\pi}{2}\,d\varphi = \frac{\pi}{2}.$$

(4) $\int_\Gamma \mathbb{A} \cdot d\mathbf{r} = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) \mathbb{A} \cdot d\mathbf{r} = -1 + 1 + \frac{\pi}{2} = \frac{\pi}{2}.$

Anm. Man kan tillämpa Stokes sats:

$$\operatorname{rot} \mathbb{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & \sin\theta & r\sin^2\theta \end{vmatrix} =$$

$$= \frac{1}{r^2\sin\theta}\left((2r\sin\theta\cos\theta)\hat{r} - (r\sin^2\theta)\hat{\theta} + (r\sin\theta\cos\theta)\hat{\varphi}\right) =$$

$$= \frac{2}{r}\cos\theta\,\hat{r} - \frac{1}{r}\sin\theta\,\hat{\theta} + \frac{1}{r}\cos\theta\,\hat{\varphi} \quad (r = 1 \text{ på } S).$$

$$\oint_\Gamma \mathbb{A} \cdot d\mathbf{r} = \iint_\Delta 2\cos\theta \cdot \sin\theta\,d\theta d\varphi = \int_0^{\pi/2} 2\cos\theta\sin\theta\,d\theta \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{2}\left[\sin^2\theta\right]_0^{\pi/2} = \frac{\pi}{2}.$$

Observera att den första oktanten svarar mot $0 \le \theta, \varphi \le \frac{\pi}{2}$, det som skrivs Δ härovan.

I en tentamen måste man avgöra tidigt vilken metod som skall användas.

Uppgift 9.20 (Sid. 13)

Lösning

$x^2+y^2+(z-1)^2=1 \Rightarrow \rho^2+(z-1)^2=1 \Leftrightarrow \rho=\sqrt{2z-z^2} \Rightarrow$

$S: r(\varphi,z)=(\sqrt{2z-z^2}\cos\varphi, \sqrt{2z-z^2}\sin\varphi, z)$; sfär.

$$\Delta: 0\le\varphi\le 2\pi,\ 0\le z\le 2.$$

$$\left.\begin{aligned} r'_\varphi &= (-\sqrt{2z-z^2}\sin\varphi, \sqrt{2z-z^2}\cos\varphi, 0) \\ r'_z &= \left(\frac{1-z}{\sqrt{2z-z^2}}\cos\varphi, \frac{1-z}{\sqrt{2z-z^2}}\sin\varphi, 1\right)\end{aligned}\right\} \Rightarrow r'_\varphi\times r'_z =$$

$$\overset{!}{=}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\varphi & \cos\varphi & 0 \\ \cos\varphi(1-z) & \sin\varphi(1-z) & \sqrt{2z-z^2}\end{vmatrix} = (\sqrt{2z-z^2}\cos\varphi)\hat{x}+$$

$$+(\sqrt{2z-z^2}\sin\varphi)+(z-1)\hat{z};\quad \hat{\rho}=(\cos\varphi,\sin\varphi,0).$$

$$A\cdot dS = A\cdot(r'_\varphi\times r'_z)\,d\varphi\,dz = \sqrt{2z-z^2}\,d\varphi dz \Rightarrow$$

$$\oiint_S A\cdot dS = \iint_\Delta \sqrt{2z-z^2}\,d\varphi dz = \int_0^2 \sqrt{1-(z-1)^2}\,dz\cdot 2\pi =$$

$$= 2\pi\int_{-1}^{1}\sqrt{1-t^2}\,dt = 2\pi\cdot\frac{\pi}{2} = \underline{\pi^2}.$$

Anm. $\overset{!}{=}$ underförstås determinantteori:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33}\end{vmatrix} = \lambda\cdot\mu\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix},\ \begin{cases}\lambda=\sqrt{2z-z^2} \\ \mu=\dfrac{1}{\sqrt{2z-z^2}}\end{cases}$$

Problem 9.21 (Sid. 13)

Lösning

$$\rho+z=2 \Leftrightarrow \sqrt{x^2+y^2}=2-z\ge 0 \Leftrightarrow \Leftrightarrow \underline{z=2-\sqrt{x^2+y^2}}$$

Jag söker kurvans projektion i xy-planet;

$$\text{z-koordinater lika} \Rightarrow z=\frac{1-y}{2}=2-\sqrt{x^2+y^2} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{x^2+y^2}=2-\frac{1-y}{2}=\frac{y+3}{2} \Leftrightarrow 4(x^2+y^2)=(y+3)^2$$

$$\Leftrightarrow 4x^2+4y^2=y^2+6y+9 \Leftrightarrow 4x^2+3y^2-6y=9 \Leftrightarrow$$

$$\Leftrightarrow 4x^2+3(y-1)^2=12 \Leftrightarrow \frac{x^2}{3}+\frac{(y-1)^2}{4}=1 \text{ (ellips).}$$

$$A=\rho\hat{\varphi}=\rho(-\sin\varphi,\cos\varphi,0)=(-\rho\sin\varphi,\rho\cos\varphi,0)=$$

$$=(-y,x,0) \Rightarrow \operatorname{rot}A=(0,0,2)$$

Jag tillämpar Stokes sats på det plana ytstycket som ligger ovanför konen och har γ till rand.

$$S: r(u,v)=\left(u,v,\frac{1-v}{2}\right),\quad D: \frac{u^2}{3}+\frac{(v-1)^2}{4}\le 1.$$

$$r'_u\times r'_v=(1,0,0)\times\left(0,1,-\frac{1}{2}\right)=\left(0,\frac{1}{2},1\right);$$

$$\oint_\Gamma A\cdot dr = \iint_S (0,0,2)\cdot\left(0,\frac{1}{2},1\right)du\,dv = 2\iint_S du\,dv =$$

$$= \left[\begin{array}{l|l|l} u=\sqrt{3}r\cos\theta & 0\le r\le 1 & \\ v=1+2r\sin\theta & 0\le\theta\le 2\pi & dudv \longrightarrow 2\sqrt{3}r\,dr\,d\theta \end{array} \right] =$$

$$= 2\int_0^1 2\sqrt{3}r\,dr \int_0^{2\pi} d\theta = 2\sqrt{3}\cdot 2\pi = \underline{4\sqrt{3}\pi}.$$

Uppgift 9.22 (Sid. 13)

Lösning

a) $u = u(x,y,z) = x^2-y^2 \Rightarrow \nabla u = (2x,-2y,0) \Rightarrow |\nabla u| =$

$= 2\sqrt{x^2+y^2} \Rightarrow \hat{u} = \dfrac{\nabla u}{|\nabla u|} = \dfrac{1}{\sqrt{x^2+y^2}}(x,-y,0).$

$v = v(x,y,z) = xy \Rightarrow \nabla v = (y,x,0) \Rightarrow \hat{v} = \dfrac{1}{\sqrt{x^2+y^2}}(y,x,0).$

$w = w(x,y,z) = z \Rightarrow \nabla w = (0,0,1) = \hat{w}.$

$\hat{u}\cdot\hat{v} = \hat{u}\cdot\hat{w} = \hat{v}\cdot\hat{w} = 0 \Rightarrow \underline{\hat{u},\hat{v},\hat{w} \text{ ortogonala}}.$

b) Skalfaktorerna är $h_u = \sqrt{x^2+y^2}$, $h_v = \sqrt{x^2+y^2}$ och $h_w = 1$. Dessa uttrycks i u,v,w.

$$\begin{cases} u = x^2-y^2 \\ v = xy \end{cases} \Leftrightarrow u^2+(2v)^2 = (x^2+y^2)^2 \Leftrightarrow \underline{x^2+y^2 = \sqrt{u^2+4v^2}}$$

Med hjälp av funktionalmatriser fås

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)^{-1} \Leftrightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} =$$

$$= \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2(x^2+y^2)} \begin{bmatrix} x & 2y & 0 \\ -y & 2x & 0 \\ 0 & 0 & 2(x^2+y^2) \end{bmatrix};$$

$\dfrac{\partial r}{\partial u} = \dfrac{\partial(x,y,z)}{\partial u} = \dfrac{1}{2(x^2+y^2)}(x,-y,0)$ och p.s.s.

$\dfrac{\partial r}{\partial v} = \dfrac{1}{2(x^2+y^2)}(2y,2x,0)$, $\dfrac{\partial r}{\partial w} = (0,0,1).$

Vi får avbildningens skalfaktorer:

$h_u = \left|\dfrac{\partial r}{\partial u}\right| = \dfrac{\sqrt{x^2+y^2}}{2(x^2+y^2)} = \dfrac{1}{2\sqrt[4]{u^2+4v^2}}$, h_v

$h_v = \left|\dfrac{\partial r}{\partial v}\right| = \dfrac{1}{\sqrt{x^2+y^2}} = \dfrac{1}{\sqrt[4]{u^2+4v^2}}$ och $h_w = 1.$

$$\operatorname{div} A = \frac{1}{h_u h_v h_w}\cdot\left[\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial v}(h_u h_w A_v) + \right.$$

$$\left. + \frac{\partial}{\partial w}(h_u h_v A_w)\right] =$$

$$= 2\sqrt{u^2+4v^2}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt[4]{u^2+4v^2}}A_u\right) + \right.$$

$$+ \frac{\partial}{\partial v}\left(\frac{1}{2\sqrt[4]{u^2+4v^2}}A_v\right) +$$

$$\left. + \frac{\partial}{\partial w}\left(\frac{1}{2\sqrt{u^2+4v^2}}A_w\right)\right].$$

Man ska hålla sig borta från origo.

Uppgift 9.23 (Sid. 13)

Lösning $x = uv\cos\varphi,\ y = uv\sin\varphi,\ z = \frac{u^2-v^2}{2}$.

a) $x^2+y^2 = (uv)^2 \Leftrightarrow uv = \sqrt{x^2+y^2}$

$$\left.\begin{aligned} u^2-v^2 &= 2z \\ uv &= \sqrt{x^2+y^2} \end{aligned}\right\} \Rightarrow (u^2+v^2)^2 = (u^2-v^2)^2 + 4u^2v^2 =$$

$$= 4z^2 + 4(x^2+y^2) = 4(x^2+y^2+z^2) \Leftrightarrow \underline{u^2+v^2 = 2\sqrt{x^2+y^2+z^2}}$$

$$\begin{cases} u^2-v^2 = 2z \\ u^2+v^2 = 2\sqrt{x^2+y^2+z^2} \\ \tan\varphi = y/x \end{cases} \Leftrightarrow \begin{cases} u^2 = \sqrt{x^2+y^2+z^2}+z \\ v^2 = \sqrt{x^2+y^2+z^2}-z \\ \varphi = \arctan y/x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \underline{u = \sqrt{\sqrt{x^2+y^2+z^2}+z}},\ \underline{v = \sqrt{\sqrt{x^2+y^2+z^2}-z}},\ \underline{\varphi = \operatorname{atn}\tfrac{y}{x}}.$$

$\underline{\Delta: 0 \le u < \infty,\ 0 \le v < \infty,\ 0 \le \varphi \le 2\pi}$.

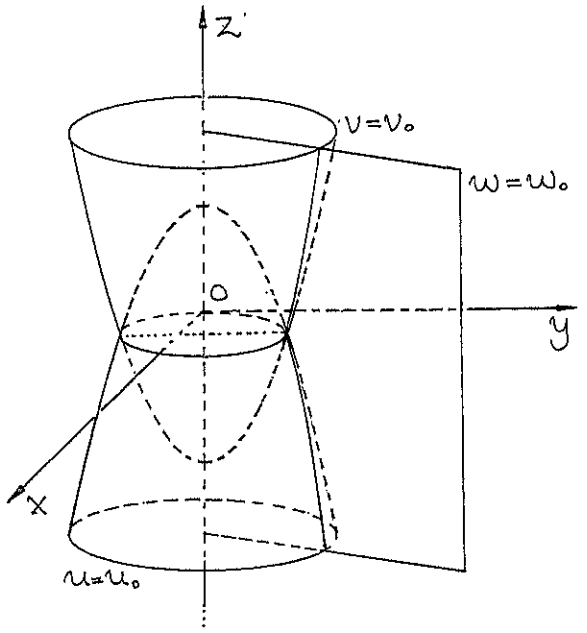
b) $\underline{u = u_0} \Rightarrow x = u_0 v\cos\varphi,\ y = u_0 v\sin\varphi,\ z = \frac{u_0^2 - v^2}{2}$

$x^2+y^2 = u_0^2 v^2 \Rightarrow z = \frac{1}{2}\left(u_0^2 - \frac{x^2+y^2}{u_0^2}\right)$ paraboloid.

$\underline{v = v_0} \Rightarrow x = uv_0\cos\varphi,\ y = uv_0\sin\varphi,\ z = \frac{u^2-v_0^2}{2}$

$x^2+y^2 = u^2v_0^2 \Rightarrow z = \frac{1}{2}\left(\frac{x^2+y^2}{v_0^2} - v_0^2\right)$ paraboloidez.

$\underline{\varphi = \varphi_0} \Rightarrow y = \tan\varphi_0 \cdot x$, planskara, (halvplan).



$\underline{u = u_0},\ \underline{v = v_0}$: Skärningen mellan två rotationsparaboloider är en cirkel (se figur).

$\underline{u = u_0},\ \underline{\varphi = \varphi_0}$: Skärningen mella en rotationsparaboloid och ett halvplan är en parabelbåge.

$\underline{v = v_0},\ \underline{\varphi = \varphi_0}$: Detsamma som ovan men med parabelbågen omkastad.

c) $\underline{r(u,v,\varphi) = \left(uv\cos\varphi,\ uv\sin\varphi,\ \frac{u^2-v^2}{2}\right)}$

$$\begin{cases} \frac{\partial \mathbf{r}}{\partial u} = (v\cos\varphi, v\sin\varphi, u) \\ \frac{\partial \mathbf{r}}{\partial v} = (u\cos\varphi, u\sin\varphi, -v) \\ \frac{\partial \mathbf{r}}{\partial \varphi} = (-uv\sin\varphi, uv\cos\varphi, 0) \end{cases} \Rightarrow \begin{cases} \frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial v} = uv - uv = 0 \\ \frac{\partial \mathbf{r}}{\partial \varphi}\cdot\frac{\partial \mathbf{r}}{\partial u} = \frac{uv^2}{2}(\underbrace{\sin 2\varphi - \sin 2\varphi}_{=0}) \\ \frac{\partial \mathbf{r}}{\partial \varphi}\cdot\frac{\partial \mathbf{r}}{\partial v} = \frac{u^2 v}{2}(\underbrace{\sin 2\varphi - \sin 2\varphi}_{=0}) \end{cases}$$

$\Rightarrow$ de paraboliska koordinaterna är ortogonala.

d) $h_u = h_v = \left|\frac{\partial \mathbf{r}}{\partial u}\right| = \left|\frac{\partial \mathbf{r}}{\partial v}\right| = \sqrt{u^2+v^2}$, $h_\varphi = \left|\frac{\partial \mathbf{r}}{\partial \varphi}\right| = uv$.

$$\operatorname{grad}\phi = \frac{1}{h_u}\frac{\partial \phi}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial \phi}{\partial v}\hat{v} + \frac{1}{h_\varphi}\frac{\partial \phi}{\partial \varphi}\hat{\varphi} =$$

$$= \frac{1}{\sqrt{u^2+v^2}}\frac{\partial \phi}{\partial u}\hat{u} + \frac{1}{\sqrt{u^2+v^2}}\frac{\partial \phi}{\partial v}\hat{v} + \frac{1}{uv}\frac{\partial \phi}{\partial \varphi}\hat{\varphi}.$$

e) $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial \varphi}$ samt skalfaktorer ser du ovan.

$$\begin{cases} \hat{u} = \frac{1}{h_u}\cdot\frac{\partial \mathbf{r}}{\partial u} = \frac{1}{\sqrt{u^2+v^2}}\frac{\partial \mathbf{r}}{\partial u} \\ \hat{v} = \frac{1}{h_v}\cdot\frac{\partial \mathbf{r}}{\partial v} = \frac{1}{\sqrt{u^2+v^2}}\frac{\partial \mathbf{r}}{\partial v} \\ \hat{\varphi} = \frac{1}{h_\varphi}\cdot\frac{\partial \mathbf{r}}{\partial \varphi} = \frac{1}{uv}\frac{\partial \mathbf{r}}{\partial \varphi} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{\varphi} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{v\cos\varphi}{\sqrt{u^2+v^2}} & \frac{v\sin\varphi}{\sqrt{u^2+v^2}} & \frac{u}{\sqrt{u^2+v^2}} \\ \frac{u\cos\varphi}{\sqrt{u^2+v^2}} & \frac{u\sin\varphi}{\sqrt{u^2+v^2}} & \frac{-v}{\sqrt{u^2+v^2}} \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}}_{\text{ON-matris}} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{v\cos\varphi}{\sqrt{u^2+v^2}} & \frac{u\cos\varphi}{\sqrt{u^2+v^2}} & -\sin\varphi \\ \frac{v\sin\varphi}{\sqrt{u^2+v^2}} & \frac{u\sin\varphi}{\sqrt{u^2+v^2}} & \cos\varphi \\ \frac{u}{\sqrt{u^2+v^2}} & \frac{-v}{\sqrt{u^2+v^2}} & 0 \end{bmatrix}}_{\text{transponatet}} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{\varphi} \end{bmatrix}$$

Basbytet mellan två ON-baser förmedlas av en ON-matris.

Uppgift 9.24 (Sid. 13)

Lösning

$\mathbf{r}(\xi, \eta, z) = (\cosh\xi\cos\eta, \sinh\xi\sin\eta, z)$; $\begin{cases} 0 \le \xi < \infty \\ -\pi < \eta \le \pi \\ -\infty < z < \infty \end{cases}$.

$$\begin{cases} \frac{\partial \mathbf{r}}{\partial \xi} = (\sinh\xi\cos\eta, \cosh\xi\sin\eta, 0) \\ \frac{\partial \mathbf{r}}{\partial \eta} = (-\cosh\xi\sin\eta, \sinh\xi\cos\eta, 0) \\ \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1) \end{cases}$$

$$\frac{\partial \mathbf{r}}{\partial \xi}\cdot\frac{\partial \mathbf{r}}{\partial \eta} = -\cosh\xi\sinh\xi\sin\eta\cos\eta + \cosh\xi\sinh\xi\cdot$$

$$\cdot\sin\eta\cos\eta = 0.$$

Att $\frac{\partial \mathbf{r}}{\partial \xi}\cdot\frac{\partial \mathbf{r}}{\partial z} = \frac{\partial \mathbf{r}}{\partial \eta}\cdot\frac{\partial \mathbf{r}}{\partial z} = 0$ är inte svår att inse; systemet är således ortogonalt.

$$h_\xi^2 = \frac{\partial r}{\partial \xi}\cdot\frac{\partial r}{\partial \xi} = (\sinh\xi\cos\eta)^2 + (\cosh\xi\sin\eta)^2 =$$

$$= \sinh^2\xi\cos^2\eta + \cosh^2\xi\sin^2\eta =$$

$$= \sinh^2\xi\cos^2\eta + (1+\sinh^2\xi)\sin^2\eta =$$

$$= \sinh^2\xi(\cos^2\eta+\sin^2\eta) + \sin^2\eta =$$

$$= \sinh^2\xi + \sin^2\eta \Leftrightarrow \underline{h_\xi = \sqrt{\sinh^2\xi+\sin^2\eta}}.$$

$$h_\eta^2 = \frac{\partial r}{\partial \eta}\cdot\frac{\partial r}{\partial \eta} = (-\cosh\xi\sin\eta)^2 + (\sinh\xi\cos\eta)^2 =$$

$$= \cosh^2\xi\sin^2\eta + \sinh^2\xi\cos^2\eta =$$

$$= (1+\sinh^2\xi)\sin^2\eta + \sinh^2\xi\cos^2\eta =$$

$$= \sin^2\eta + \sinh^2\xi(\sin^2\eta+\cos^2\eta) =$$

$$= \sin^2\eta + \sinh^2\xi = 1-\cos^2\eta+\sinh^2\xi =$$

$$= \cosh^2\xi - \cos^2\eta \Leftrightarrow \underline{h_\eta = \sqrt{\cosh^2\xi-\cos^2\eta}}.$$

$h_z = 1$ är lätt att visa.

$$\xi = \xi_0 \Rightarrow x = \cosh\xi_0\cos\eta \wedge y = \sinh\xi_0\sin\eta \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{x}{\cosh\xi_0}\right)^2 + \left(\frac{y}{\sinh\xi_0}\right)^2 = 1. \qquad (1)$$

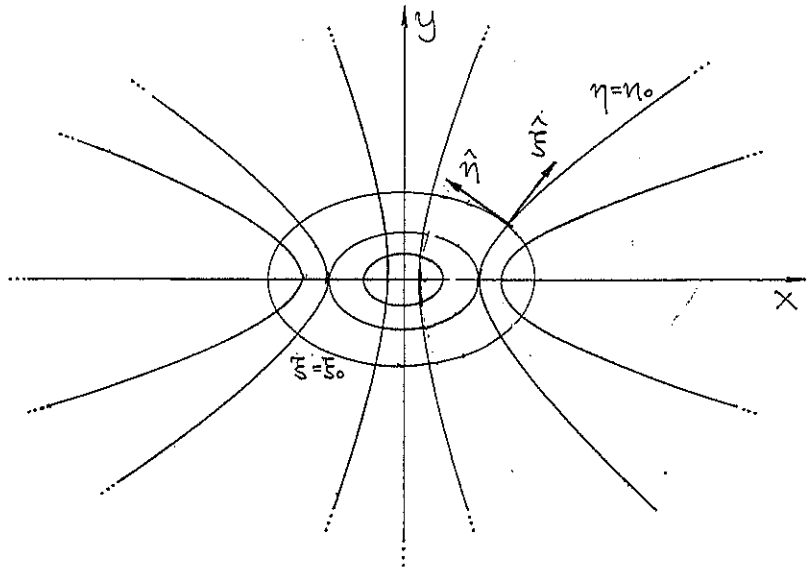
η- koordinatkurvorna är ellipser.

$$\eta = \eta_0 \Rightarrow x = \cosh\xi\cos\eta_0 \wedge y = \sinh\xi\sin\eta_0 \Rightarrow$$

$$\Rightarrow \left(\frac{x}{\cos\eta_0}\right)^2 - \left(\frac{y}{\sin\eta_0}\right)^2 = 1 \qquad (2)$$

ξ- koordinatkurvorna är hyperbler.

Anm. I (1) underförstås den trigonometriska ettan $\cos^2\eta+\sin^2\eta=1$ och i (2) underförstås den hyperboliska ettan $\cosh^2\xi-\sinh^2\xi=1$.



Uppgift 9.25 (Sid. 13)

Lösning

$$r(R,\theta,\varphi) = ((a-R\cos\theta)\cos\varphi,\ (a-R\cos\theta)\sin\varphi,\ R\sin\theta)$$

$$\left.\begin{aligned}\frac{\partial r}{\partial R} &= (-\cos\theta\cos\varphi, -\cos\theta\sin\varphi, \sin\theta)\\ \frac{\partial r}{\partial\theta} &= (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)\\ \frac{\partial r}{\partial\varphi} &= ((R\cos\theta-a)\sin\varphi, (a-R\cos\theta)\cos\varphi, 0)\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \frac{\partial r}{\partial R}\cdot\frac{\partial r}{\partial\theta} = -R\sin\theta\cos\theta\cos^2\varphi - R\sin\theta\cos\theta\sin^2\theta +$$

$$+ R\sin\theta\cos\theta =$$

$$= -R\sin\theta\cos\theta(\cos^2\varphi+\sin^2\varphi) + R\sin\theta\cos\theta =$$

$$= -R\sin\theta\cos\theta + R\sin\theta\cos\theta = 0 \text{ och p.s.s.}$$

$$\frac{\partial r}{\partial R}\cdot\frac{\partial r}{\partial\varphi} = (a-R\cos\theta)(\cos\theta\sin\varphi\cos\varphi - \cos\theta\sin\varphi\cos\varphi) = 0$$

och $\frac{\partial r}{\partial\theta}\cdot\frac{\partial r}{\partial\varphi} = (a-R\cos\theta)(-\sin\theta\sin\varphi\cos\varphi + \sin\theta\sin\varphi\cos\varphi) = 0.$

$$h_1 = \left|\frac{\partial r}{\partial R}\right| = (\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta)^{1/2} =$$

$$= (\cos^2\theta(\cos^2\varphi+\sin^2\varphi) + \sin^2\theta)^{1/2} =$$

$$= (\cos^2\theta+\sin^2\theta)^{1/2} = 1.$$

$$h_2 = \left|\frac{\partial r}{\partial\theta}\right| = R(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta)^{1/2} =$$

$$= R(\sin^2\theta(\cos^2\varphi+\sin^2\varphi) + \cos^2\varphi)^{1/2} =$$

$$= R(\sin^2\theta+\cos^2\theta)^{1/2} = R.$$

$$h_3 = \left|\frac{\partial r}{\partial\varphi}\right| = (a-R\cos\theta)(\sin^2\varphi+\cos^2\varphi)^{1/2} = a-R\cos\theta.$$

Uppgift 9.26 (Sid. 13)

Lösning

$$\operatorname{grad}\Omega = \frac{\partial}{\partial r}\Omega\,\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\Omega\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\Omega\hat{\varphi}$$

$$r\cdot\operatorname{grad}\Omega = n\Omega \Leftrightarrow r\frac{\partial\Omega}{\partial r} = n\Omega \Leftrightarrow r^n\frac{\partial\Omega}{\partial r} - nr^{n-1}\Omega = 0$$

$$\Leftrightarrow \frac{\partial}{\partial r}\left(\frac{\Omega}{r^n}\right) = 0 \Leftrightarrow \frac{\Omega}{r^n} = f(\theta,\varphi) \Leftrightarrow \underline{\Omega = r^n f(\theta,\varphi)}.$$

Uppgift 9.27 (Sid. 13)

Lösning

$$\nabla = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{z}\frac{\partial}{\partial z}; \qquad \mathbf{F} = \rho\cos\varphi\,\hat{\rho} + \sin\varphi\,\hat{\varphi};$$

$$\frac{\partial}{\partial\varphi}\hat{\rho} = \frac{\partial}{\partial\varphi}(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}) = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} = \hat{\varphi};$$

$$\frac{\partial}{\partial\varphi}\hat{\varphi} = \frac{\partial}{\partial\varphi}(-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}) = -\cos\varphi\,\hat{x} - \sin\varphi\,\hat{y} = -\hat{\rho};$$

$$\mathbf{F}\cdot\nabla = \rho\cos\varphi\frac{\partial}{\partial\rho} + \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi};$$

$$(\mathbf{F}\cdot\nabla)\mathbf{F} = \left(\rho\cos\varphi\frac{\partial}{\partial\rho} + \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)(\rho\cos\varphi\,\hat{\rho} + \sin\varphi\,\hat{\varphi}) =$$

$$= \rho\cos\varphi\frac{\partial}{\partial\rho}(\rho\cos\varphi\,\hat{\rho}) + \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}(\rho\cos\varphi\,\hat{\rho}) +$$

$$+ \rho\cos\varphi\frac{\partial}{\partial\rho}(\sin\varphi\hat{\varphi}) + \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}(\sin\varphi\,\hat{\varphi}) =$$

$$= r\cos\varphi\cdot\cos\varphi\,\hat{\rho} + \frac{\sin\varphi}{\rho}(-\rho\sin\varphi\hat{\rho} + \rho\cos\varphi\,\hat{\varphi}) +$$

$$+\frac{\sin\varphi}{\rho}(\cos\varphi\hat{\varphi}-\sin\varphi\hat{\rho}) =$$

$$= \rho\cos^2\varphi\,\hat{\rho} - \sin^2\varphi\hat{\rho} + \sin\varphi\cos\varphi\,\hat{\varphi} + \frac{\sin\varphi\cos\varphi}{\rho}\hat{\varphi} -$$

$$-\frac{\sin^2\varphi}{\rho}\hat{\rho} =$$

$$= (\rho\cos^2\varphi - \sin^2\varphi - \frac{\sin^2\varphi}{\rho})\hat{\rho} + \sin\varphi\cos\varphi(1+\frac{1}{\rho})\hat{\varphi} =$$

$$= (\rho - \rho\sin^2\varphi - \sin^2\varphi - \frac{1}{\rho}\sin^2\varphi)\hat{\rho} + \sin\varphi\cos\varphi(1+\frac{1}{\rho})\hat{\varphi} =$$

$$= \underline{(\rho - (\rho+1+\frac{1}{\rho})\sin^2\varphi)\hat{\rho} + (1+\frac{1}{\rho})\sin\varphi\cos\varphi\,\hat{\varphi}}.$$